Amplify Math

Algebra 1

Volume 2: Units 4–6

Student Edition

About Amplify

Amplify is dedicated to collaborating with educators to create learning experiences that are rigorous and riveting for all students. Amplify creates K–12 core and supplemental curriculum, assessment, and intervention programs for today's students.

A pioneer in K–12 education since 2000, Amplify is leading the way in next-generation curriculum and assessment. All of our programs provide teachers with powerful tools that help them understand and respond to the needs of every student.

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Hello, curious mind!

Welcome to Algebra 1. The word “algebra” originates from the Arabic word “al-jabr,” which means “the reunion of broken parts.” Don’t worry, you’ll stay in one piece. But this is a year to break up the mathematics, determine what you know, and develop strategies to solve all sorts of problems.

You see, this year, you will break the constraints (and conventions) of homecoming, plan for life after high school, overcome “choice overload,” and strategize your fundraising efforts to support the environment. And that’s just Unit 1!

We promise you, it’s all possible. This year, you’ll see not only that math makes sense, but it can be fun, too.

Before you dig in, we want you to know two things:

This book is special! It’s where you’ll record your thoughts, strategies, explanations, and, occasionally, any award-winning doodles that might come to mind.

When you go online, you won’t be mindlessly plugging numbers into your device . . . You’ll be pushing, pulling, crawling, teleporting, melting . . . , well, let’s just say you’ll be doing a lot of things, and you’ll be teaming up with your classmates as you do.

Let’s dig in!

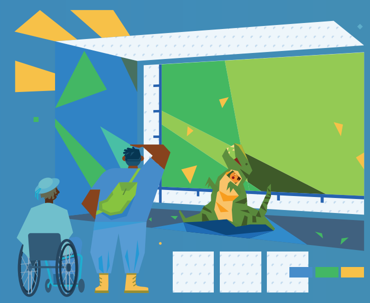
Sincerely,
The Amplify Math Team

Volume 1

Unit 1 Linear Equations, Inequalities, and Systems

In this unit, you will encounter a range of situations, from high school to adulthood. Along the way, you will discover how constraints connect to equations and inequalities, and how they can be used to help with decision-making.

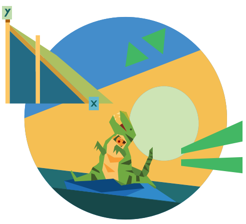
Unit Narrative:
Adulting (Making Life Decisions)



How did a tragic accident end a three-month strike?
Revisit how equations and inequalities can be used to model real-world situations, and how they can help you make decisions.

How do first-gen Americans vault the hurdles of college?
"Solving" an equation doesn't always mean finding an unknown value — sometimes it can mean changing the equation's very structure.

What's after high school?
Whether you work, intern, attend college, or do something else, inequalities can be used to model your time and money, helping you manage both of them.



Are you a 'Boomerang-er'?
For better or for worse, life is full of constraints. Discover new strategies for solving problems with multiple constraints, which you will see time and again.

Is there such a thing as too much choice?
What happens when the decisions become more complicated? Look at the big picture and then fine-tune where the decision overlaps.

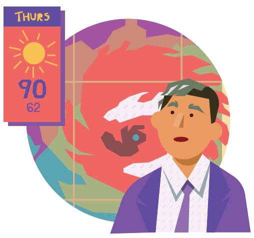
Unit 2 Data Analysis and Statistics

Unit Narrative:
Analyzing Climate Change

In this unit, you will explore data sets with one or two variables, often related to one of the most pressing threats we face as humanity: climate change. Along the way, you will encounter new statistical measures of center, spread, and association.

How can we protect ourselves from a zombie virus?

Remember dot plots, histograms, and box plots? Revisit them through temperature data, while describing the data's center and variability.



Is Sandy the new normal?

Meet the most commonly used measure of variability: standard deviation.

What is "Day Zero"?

You've seen linear models before, but now you will (finally!) see how to identify the "best" model, using what are called residuals.

What makes storms worse and has nothing to do with the weather?

Use two-way tables to see how the changing climate has affected marginalized people around the world.

Who is the "water warrior"?

Finally, you can say just how strong or weak an association is using a new statistical measure: the correlation coefficient.

Unit 3 Functions and Their Graphs

Unit Narrative:
Artscapes

You will expand your understanding of functions, their representations and graphs. Along the way, you will write, graph, and interpret a variety of functions and their inverses.

How did the blues find a home in Memphis?
Remember representing functions with description, tables, graphs, and equations? Take another look as you visit Memphis, and meet another common tool to represent and interpret functions: *function notation*.

What's the function of a jazz solo?
The way you describe a graph helps you gain insight on the relationship it represents. *Average rate of change*, *domain*, and range help to construct and interpret graphs more precisely.

Where did the world meet soul?
Piecewise functions allow you to represent a relationship between two quantities as a set of rules. The absolute value function is a specific piecewise function that represents the distance from zero.

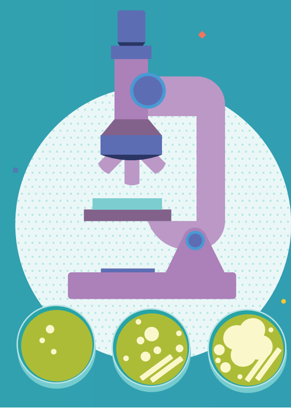
How do you get Sunday shoppers to hear your song?
What happens if you reverse the process used in a relationship between two quantities? What changes? What stays the same? You can explore these questions using inverses of functions.

Volume 2

Unit 4 Introducing Exponential Functions

This is a unit of mathematical discovery, where relationships between quantities are unlike any function you have seen up to this point. You will encounter the explosiveness of exponential growth, as well as the lingering of exponential decay, through the lenses of infectious disease, vaccination, and prescription drug costs.

Unit Narrative:
Infectious Diseases, Vaccines, and Costs



Where do baby bacteria come from?

Examine nonlinear functions using tables and graphs, before defining an exponential relationship. You'll represent exponential growth using an equation and explore it in context.

How did an enslaved person save the city of Boston?

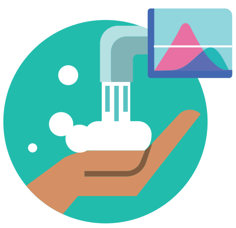
Examine growth factors between 0 and 1, as you develop an understanding of exponential decay.

What does growing or shrinking look like on a graph?

Identify exponential relationships as exponential functions, and whether a graph is discrete.

Want to be CEO for a day?

Make sense of repeated percent increase and see how it relates to compound interest.



How does distance make the curve grow flatter?

Compare the growth of different kinds of functions and finish with an exploration of how social distancing can combat the dangers of an epidemic.

Unit 5 Introducing Quadratic Functions

Unit Narrative:
Squares in Motion

What goes up must come down. In this unit, you will study quadratic functions. By analyzing and comparing patterns, tables, graphs, and equations, you will gain an appreciation for the special features of quadratic functions and the situations they represent.

Unit 6 Quadratic Equations

In this unit, you will explore how people have learned to solve quadratic equations throughout history. You will write, solve, and explore strategies for solving quadratic equations.

Unit Narrative:
The Evolution of Solving Quadratic Equations

LAUNCH

Sub-Unit 1 Connecting Quadratic Functions to Their Equations 913

How did the Nile River spur on Egyptian mathematics?
Revisit projectile motion and maximizing revenue, as you discover new meanings for the zeros of a quadratic function.

Sub-Unit 2 Factoring Quadratic Expressions and Equations 941

When is zero more than nothing?
Understand the importance of zero when solving quadratic equations. Then, efficiently factor quadratics using a variety of strategies.

Sub-Unit 3 Completing the Square 979

How many ways can you crack an egg?
Discover the ancient art of taking a quadratic expression and completing the square. It's all about that missing piece.

Sub-Unit 4 Roots and Irrationals 1019

Where does a number call its home?
Subtraction and division took you from whole numbers to rationals. Now you must look beyond them as you operate with irrational numbers.

What was the House of Wisdom?

Discover strategies for solving any quadratic equation. You will also determine which strategies are more efficient.

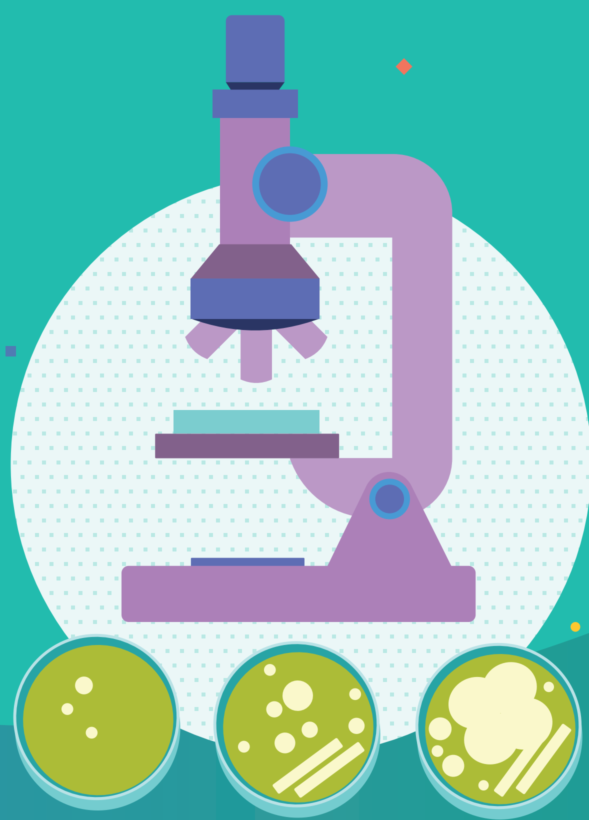
UNIT 4

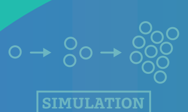
Introducing Exponential Functions

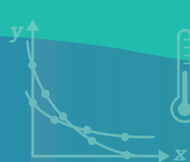
This is a unit of mathematical discovery, where relationships between quantities are unlike any function you have seen up to this point. You will encounter the explosiveness of exponential growth, as well as the lingering of exponential decay, through the lenses of infectious disease, vaccination, and prescription drug costs.

Essential Questions

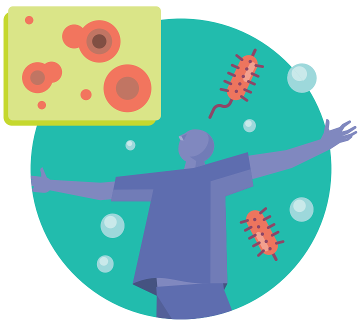
- What characterizes exponential growth and decay?
- What are real-world models of exponential growth and decay?
- How can you differentiate exponential growth from linear growth, given a real-world data set?
- *(By the way, can one person infect the entire world?)*







Ten Americans sign a pledge to support the environment. That number doubles every day. After how many days will every American have signed the pledge?



SUB-UNIT

1 Looking at Growth

Narrative: From bacteria growth to social media, explore a special kind of nonlinear growth.

You'll learn . . .

- how exponential growth differs from linear growth.

SUB-UNIT

2 A New Kind of Relationship

Narrative: Exponential relationships can help explain how smallpox was eradicated.

You'll learn . . .

- about growth factors that represent exponential decay.

SUB-UNIT

3 Exponential Functions

Narrative: Exponential functions help us track the spread of disease and effects of medication.

You'll learn . . .

- about exponential growth and decay as *functions*.

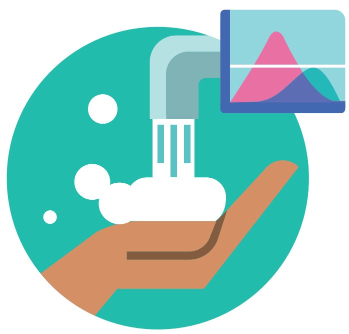
SUB-UNIT

4 Percent Growth and Decay

Narrative: The response to medication can be modeled with percent change and exponential expressions.

You'll learn . . .

- how *growth factors* differ from *growth rates*.



SUB-UNIT

5 Comparing Linear and Exponential Functions

Narrative: Exponential functions help us understand the spread of disease — and how to slow it down.

You'll learn . . .

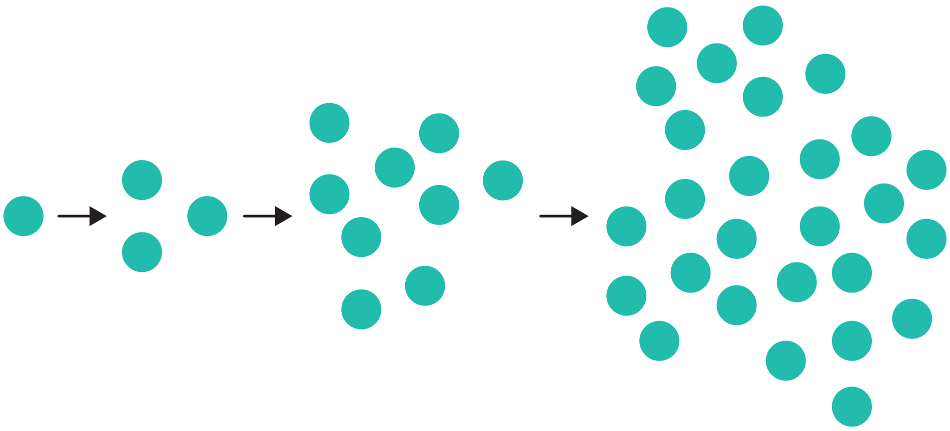
- whether linear or exponential functions grow at a faster rate.

What Is an Epidemic?

Let's look at some growth.

Warm-up Notice and Wonder

Consider the following image. What do you notice? What do you wonder?



1. I notice . . .

2. I wonder . . .

Collect and Display: As you share what you notice and wonder, your teacher will collect the language you use and add it to a class display. Continue to add to this display throughout this unit.

Log in to Amplify Math to complete this lesson online.

Name: ________________ Date: ________ Period: ________

Activity 1 Simulating an Epidemic

Record the data from the class simulation.

1. Complete a data table for each simulation, so that it shows the total number of infected students after each round. (Some cells in the tables may be left empty, depending on how many students are in your class.)

Simulation 1:

x						
y						

Simulation 2:

x						
y						

Simulation 3:

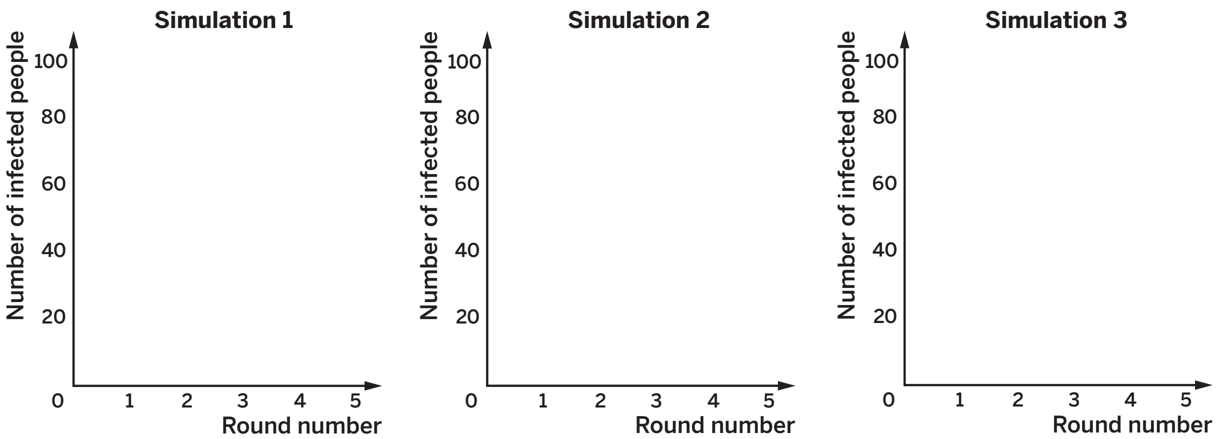
x						
y						

2. What patterns do you notice?

3. How did you define your variables x and y?

Activity 1 Simulating an Epidemic (continued)

4. Create a graph for each of your tables.



5. For each simulation, determine the number of rounds you think are needed until $1{,}000$ students would be infected.

6. Are these graphs linear or nonlinear? Explain your thinking.

Are you ready for more?

Suppose a simulation was conducted in a school with $2{,}000$ students, and started with 4 infected students. Every round, each infected student infects one other student. How many students would be infected after Round 8? After Round 9? Use drawings, models, or words to explain your thinking.

Name: .... Date: .... Period: ....

Activity 2 Real-World Implications

1. Predict how your graphs would change if every sick person infected 2 new people each round (rather than just 1). What do you think would happen to the shape of the graph?

2. In the real world, infected people might come in contact with other infected people (rather than always infecting someone new). How might this change the outcome of the simulation?

3. In the real world, some people might be immune. Predict how this could change the outcome of the simulation.

4. Name other real-world phenomena that could result in these types of patterns.

STOP

Unit 4 Introducing Exponential Functions

Infectious Diseases, Vaccines, and Costs

In January of 2020, the first cases of the novel coronavirus, COVID-19, were reported in the United States. Many officials and some of the American public were worried about this disease. The disease appeared to be 20 times more lethal than the seasonal flu, while at the same time being more contagious than the common cold. Worse, there were concerns about it being spread by asymptomatic transmission, meaning if you were infected with COVID-19 but felt healthy, you could still transmit the virus to others.

Amid these concerns, most Americans were *not* worried, because not many people had been infected at the time. On February 1 of 2020, there were only eight confirmed cases in the U.S. By March 1, there were 89. Among a population of 300+ million, how could small numbers like these be so alarming?

As you saw in today's lesson, when the number of infected people keeps doubling, then doubling again, and again and again, things can get overwhelming pretty fast. By April 1, there were more than 200,000 reported cases in the U.S. And a year later, even with protective measures that encouraged social distancing and mask wearing, there were more than 30 million cases and 500,000 dead in the U.S. alone.

This explosive growth is completely unlike the functions you have seen up to this point. In these next lessons, not only will you map out this kind of growth, but you will think about how to manage and predict it. You will hear stories about the toll diseases have taken on us as a species, as well as stories of perseverance and hope — and what we can do to fight back.

Welcome to Unit 4.

Practice

Name: .... Date: .... Period: ....

1. For the two following patterns, describe how their growth is similar or different.

   a. $1, 4, 7, 10, 13, 16, 19, 22, 25, \ldots$

   b. $1, 3, 9, 27, 81, 243, \ldots$

2. A student transforms into a zombie at a school dance. The table represents what happens afterwards, where x represents the number of minutes that have passed since the first student turned into a zombie, and y is the total number of zombies at the dance.

x	0	5	10	15	20	25	30	35	40	45	50	55
y	1	3	9	27	81	243	729	732	732	732	732	732

What does this table tell you about what happened? Explain your thinking.

3. Select the input-output pairs for the function $f(x) = x^2$ that will make it true.

   A. $(-1, 1)$

   B. $(3, 9)$

   C. $(4, 8)$

   D. $(3, 6)$

   E. $\left(\frac{1}{2}, \frac{1}{4}\right)$

   F. $\left(\frac{1}{2}, 1\right)$

4. Match each function rule with its corresponding verbal description.

   a. $f(x) = 3x - 2$

   b. $g(x) = 3(x - 2)$

   c. $h(x) = 2x - 3$

   d. $h(x) = 2(x - 3)$

   .... To get the output, subtract 3 from the input, then multiply the result by 2.

   .... To get the output, multiply the input by 3, then subtract 2 from the result.

   .... To get the output, subtract 2 from the input, then multiply the result by 3.

   .... To get the output, multiply the input by 2, then subtract 3 from the result.

Practice

Name: ______ Date: ______ Period: ______

5. Solve the inequality. Show your thinking.

$5x - 10 \geq 3(2 - x) + 4$

6. Label each table as *linear* or *nonlinear*. For tables that could represent a linear function, write the function.

a

x	0	1	2	3	4	5
$f(x)$	0.5	1	1.5	2	2.5	3

b

x	0	1	2	3	4	5
$g(x)$	0.5	1.5	4.5	13.5	40.5	121.5

c

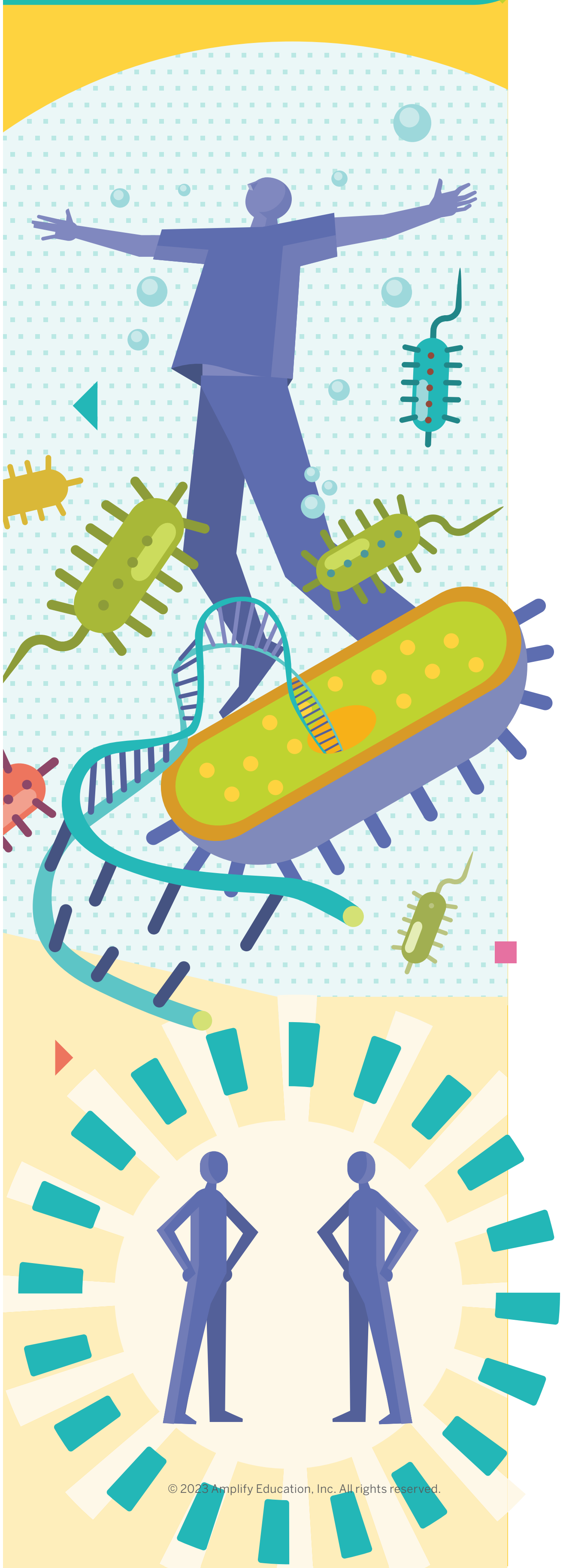
x	-4	-2	0	2	4	6
$h(x)$	-64	-8	0	8	64	216

d

x	-4	-2	0	2	4	6
$k(x)$	12	6	0	-6	-12	-18

SUB-UNIT

1 Looking at Growth



Narrative Connections

Where do baby bacteria come from?

Picture this:

You're somewhere warm and surrounded by water. You see nothing, you feel nothing. But everything is perfect. The water is perfect, the pH is perfect, and the environment is rich with nutrients.

Inside you, something happens . . .

A great burst of energy, as your DNA uncoils — tearing itself into separate strands, like a strip of leather. They swim apart, pulling and stretching against your insides.

They pull so hard that you feel your midsection tighten, like a rubber band about to snap. And then all of a sudden, *POP* — now there are two of you!

This is the process of "binary fission," and it's how bacteria grow — from one cell into two, two into four, four into eight, on and on, sometimes in a matter of minutes. The process is so dramatic that a single cell of *E. coli* can grow into a million cells in just seven hours.

The speed and magnitude of this kind of growth can be overwhelming at first. It seems different from linear growth, but how? If you take your time and look at the patterns, you can put it into perspective. With the right tools, numbers that once seemed beyond comprehension can become predictable.

Unit 4 | Lesson 2

Patterns of Growth

Let's compare different patterns of growth.

Warm-up Which One Doesn't Belong?

Which table does not belong? Explain your thinking.

Table A

x	y
1	8
2	16
3	24
4	32
8	64

Table B

x	y
0	0
2	16
4	32
6	48
8	64

Table C

x	y
0	1
1	4
2	16
3	64
4	256

Table D

x	y
0	4
1	8
2	12
3	16
4	20

Log in to Amplify Math to complete this lesson online.

Name: ______ Date: ______ Period: ______

Activity 1 Pharmacy Expansion

A retail pharmacy company has 5 locations in one state. The company is considering two plans for expanding their chain of pharmacies nationwide.

Plan A: Open 20 new pharmacies each year.
Plan B: Double the number of pharmacies each year.

1. Which plan do you think will result in a greater number of pharmacies over the next 10 years?

2. Complete the table for each plan.

Number of pharmacies

Year	Plan A	Plan B
0	5	5
1	25	10
2		
3		
4		
5		
6		
7		
8		
9		
10		

3. Compare and contrast Plans A and B. What pattern(s) did you notice as you were completing in the table from one year to the next?

Activity 1 Pharmacy Expansion (continued)

4. Does either plan have a *common difference?* If so, what is it?

5. If you know how many pharmacies there are in a certain year, how could you determine the number of pharmacies there will be 3 years later according to Plan A? Plan B?

6. Which plan will result in more pharmacies over the next 10 years? Does this match your prediction?

Are you ready for more?

Suppose the pharmacy company decides to expand from the 5 pharmacies it has now, so that it will have between 600 and 800 pharmacies 5 years from now.

1. Create a plan for the company to achieve this, so that it adds the same number of pharmacies each year.

2. Create a plan for the company to achieve this, so that the number of stores is multiplied by the same factor each year. (You may need to round the outcome to the nearest whole number for some years.)

Name: ______ Date: ______ Period: ______

Activity 2 Friends and Followers

1. Read each scenario. Then match each table or expression with its corresponding scenario.

Scenario 1: Tyler has 80 followers on The Gram. His number of followers triples each year. How many followers will he have after 4 years?

Scenario 2: Priya currently has 80 friends on a social media app. Every day, she adds 3 new friends. How many social media friends will she have after 4 days?

a. $80 \cdot 3 \cdot 3 \cdot 3 \cdot 3$

Scenario ______

b. $80 + 4 \cdot 3$

Scenario ______

c.

x	y
0	80
1	240
2	720
3	2,160
4	6,480

Scenario ______

d.

x	y
0	80
1	83
2	86
3	89
4	92

Scenario ______

e. $80 + 3 + 3 + 3 + 3$

Scenario ______

f. $80 \cdot 81$

Scenario ______

2. Which scenario represents a linear pattern? Explain your thinking.

3. Which scenario represents a nonlinear pattern? Explain your thinking.

4. Which scenario has a constant factor? Explain what the constant factor means in the scenario.

STOP

Summary

In today's lesson . . .

You looked at tables and expressions that represent two different patterns of growth.

Pattern A

x	y
1	50
2	150
3	250
4	350

- This pattern increases at a constant rate of 100. This pattern is *linear*.
- 100 is the ***common difference***. You can add 100 to any term in this pattern to find the next term. You can determine any term in the pattern by repeated addition of the *common difference*.

Pattern B

x	y
1	50
2	150
3	450
4	1350

- This pattern grows by a factor of 3. This pattern is *nonlinear*.
- 3 is the ***common factor***. You can multiply any term in this pattern by 3 to find the next term. You can determine any term in the pattern by repeated multiplication of the *common factor*.

Reflect:

Practice

Name: ______________________ Date: __________ Period: __________

1. The population of a colony of ants is $10{,}000$ at the start of April. After that, it triples each month.

   a. Complete the table.

Months since April	Ant Population
0	$10{,}000$
1	
2	
3	
4	

   b. What do you notice about the population from one month to the next?

   c. If there are n ants in one month, how many ants will there be one month later?

2. A swimming pool contains 500 gallons of water. A hose is turned on, and it fills the pool at a rate of 24 gallons per minute. Which expression represents the amount of water in the pool, in gallons, after 8 minutes?

   A. $500 \cdot 24 \cdot 8$

   B. $500 + 24 + 8$

   C. $500 + 24 \cdot 8$

   D. $500 \cdot 24^8$

3. The population of a city is $100{,}000$. It doubles each decade for 5 decades. Select *all* expressions that represent the population of the city after 5 decades.

   A. $32{,}000$

   B. $320{,}000$

   C. $100{,}000 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2$

   D. $100{,}000 \cdot 5^2$

   E. $100{,}000 \cdot 2^5$

Practice

Name: Date: Period:

4. The table shows how the depth, in centimeters, of the water in a swimming pool increases as the pool is filled.

Minutes	Depth (cm)
0	150
1	150.5
2	151
3	151.5

a Does the depth increase by the same amount each minute? Explain your thinking.

b Does the depth increase by the same factor each minute? Explain your thinking.

5. Account C starts with $10 and doubles each week. Account D starts with $1,000 and grows by $500 each week. When will Account C contain more money than Account D? Explain your thinking.

6. Match the equivalent expressions.

a $x \cdot x \cdot x \cdot x \cdot x$ .... $5x$

b 4^3 .... x^5

c $3 \cdot 3$ .... 2^6

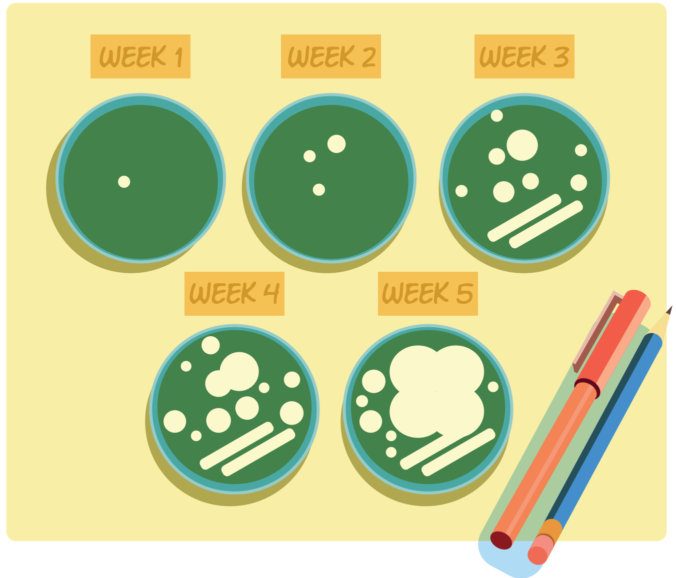
d $x + x + x + x + x$ .... 2^3

e $2 \cdot 2 \cdot 2$ .... $\frac{1}{2^6}$

f $\left(\frac{1}{2}\right)^6$ .... 3^2

Name: ______ Date: ______ Period: ______

Unit 4 | Lesson 3



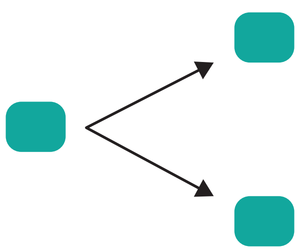
Growing and Growing

Let's connect different patterns to their tables and graphs.

Warm-up Splitting Bacteria

Under ideal conditions, *E. coli* bacteria in a laboratory petri dish divide every 20 minutes. This diagram shows a single bacterium dividing into two bacteria in the first 20 minutes.

0 min **20 min** **40 min** **60 min**



1. Complete the diagram by sketching the number of bacteria at 40 and 60 minutes. What pattern do you notice?

2. How many bacteria are there after the first hour? Explain your thinking.

Log in to Amplify Math to complete this lesson online.

Activity 1 Viral Memes

Read each scenario. Then use the scenarios to complete the problems.

Scenario 1: Andre shares a meme with 20 followers. One of his followers shares the meme with 10 more followers per hour, for the next 5 hours.

Andre's shares						
Hour	0	1	2	3	4	5
Total shares	20	30	40	50	60	70

Scenario 2: Jada shares a meme with 3 followers. Each follower shares with another 3 followers each hour. This continues for the next 5 hours.

Jada's shares						
Hour	0	1	2	3	4	5
New shares	3	9	27	81	243	729

1. Compare Andre's shares to Jada's shares. Describe the growth patterns of shares per hour.

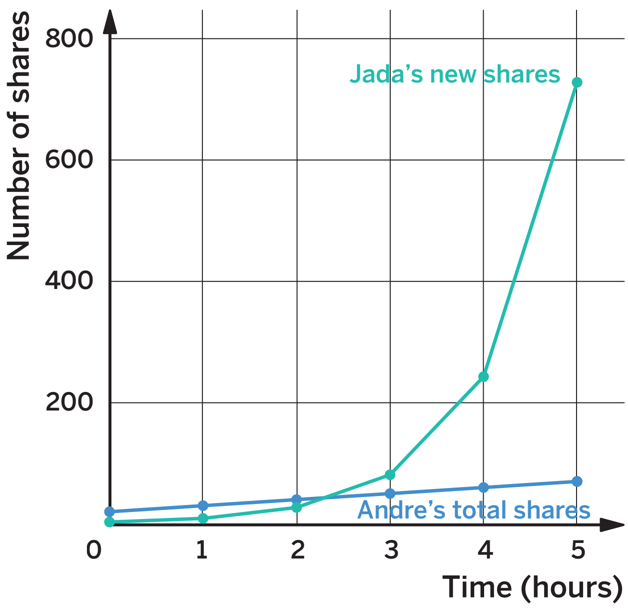
2. What is the common difference for the table that shows linear growth? What is the common factor for the table that shows nonlinear growth?

3. How many total shares will there be in both scenarios after 6 hours have passed? Explain your thinking. (Hint: Does Jada's table show the total number of shares?)

Name: ______ Date: ______ Period: ______

Activity 1 Viral Memes (continued)

The graph shows the growth patterns for Andre's and Jada's shares.



4. Compare the graphs of shares for the two memes. Do they appear to match your descriptions from Problem 1?

5. Use the graph to determine when Jada's new shares will exceed Andre's total shares.

Are you ready for more?

Use the table of values for Scenarios 1 and 2 to write a rule for each growth pattern.

Activity 2 Viral Reproduction Number

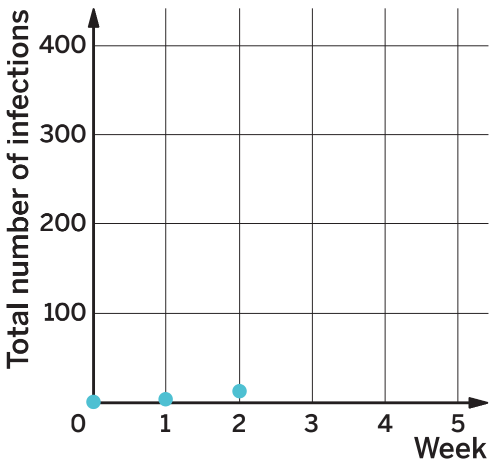
The "basic reproduction number" of an infectious disease is often referred to as R_0, or "R-nought." If $R_0 = 1$, then each infected person is likely to infect 1 new person, who will infect 1 other person, and so on. If $R_0 = 2$, then each infected person is likely to infect 2 new people, who also each infect 2 new people, and so on.

1. Measles is a particularly infectious disease. What would you guess its value of R_0 is? In other words, in a population without immunity to measles, how many more people would each infected person infect?

2. A particular strain of the flu has a basic reproduction number of $R_0 = 3$, and is infectious for a week. Complete the table and graph for the total number of infections one person with this flu strain will cause.

Scenario 1: $R_0 = 3$

Week	Number of new infections	Total number of infections
0	1	1
1	3	4
2	9	13
3		
4		
5		



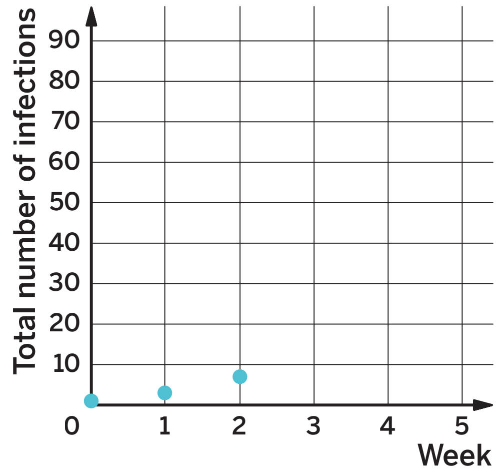
Name: ________ Date: ________ Period: ________

Activity 2 Viral Reproduction Number (continued)

3. After the Centers for Disease Control issues a warning advising people to wear masks during the outbreak, the basic reproduction number falls to $R_0 = 2$. Complete the table and graph for the total number of infections one person with this flu strain will now cause.

Scenario 2: $R_0 = 2$

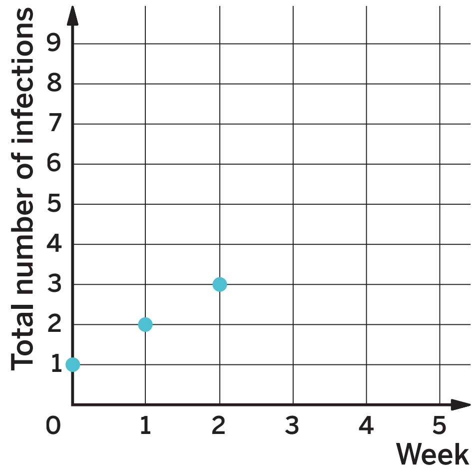
Week	Number of new infections	Total number of infections
0	1	1
1	2	3
2	4	7
3		
4		
5		



4. A vaccine for this strain of flu has been made available to the public, so that the basic reproduction number is now $R_0 = 1$. Complete the table and graph for the total number of infections one person with this flu strain will now cause.

Scenario 3: $R_0 = 1$

Week	Number of new infections	Total number of infections
0	1	1
1	1	2
2	1	3
3		
4		
5		



5. Compare the graphs of the three different scenarios. Which graphs are linear, and which are nonlinear?

STOP

Summary

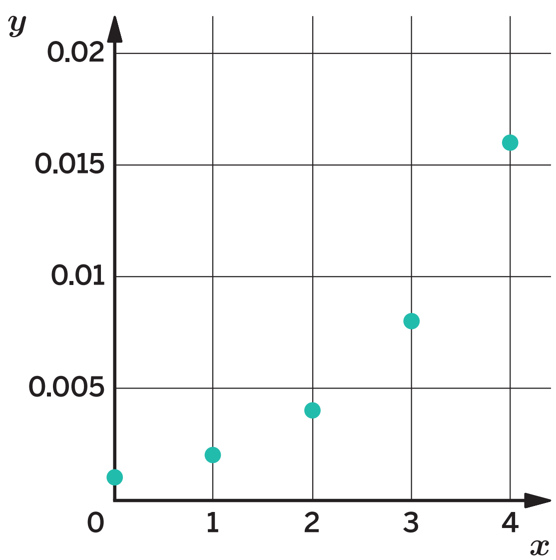
In today's lesson . . .

You compared tables and graphs of linear and nonlinear growth patterns. When you repeatedly double (or triple, or quadruple, etc.) a positive number, it soon becomes very large.

The table and graph show doubling of a small number, 0.001, using the equation $y = 0.001 \cdot 2^x$.

Expression: $0.001 \cdot 2^x$

x	0	1	2	3	4
$y = 0.001 \cdot 2^x$	0.001	0.002	0.004	0.008	0.016



Reflect:

Name: ________________ Date: ________ Period: ________

Practice

1. Which expression is equivalent to 2^7?

A. $2 + 2 + 2 + 2 + 2 + 2 + 2$

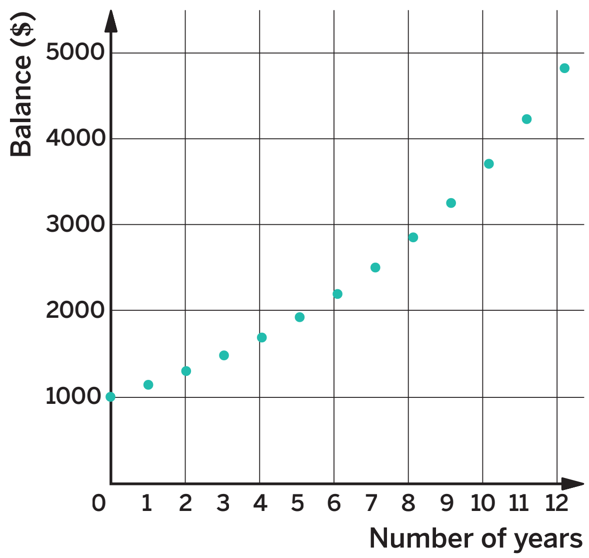
B. $2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2$

C. $2 \cdot 7$

D. $2 + 7$

2. Evaluate the expression $3 \cdot 5^x$ when x is 2.

3. The graph shows the yearly balance, in dollars, in an investment account.



a. What is the initial balance in the account?

b. Is the account growing by the same number of dollars each year? Explain your thinking.

c. A second investment account starts with $\$2,000$ and grows by $\$150$ each year. Sketch the values of this account on the graph.

d. How do the balances in each account compare?

Practice

Name: .... Date: .... Period: ....

4. Jada rewrites $5 \cdot 3^x$ as $15x$. Do you agree with Jada that these are equivalent expressions? Explain your thinking.

5. Han's account has an initial balance of $\$200$ and doubles every year. Tyler's account has an initial balance of $\$1{,}000$ and increases by $\$100$ each year.

   a. How long does it take for each account to double?

   b. When will the value of Han's account be greater than the value of Tyler's account?

   c. How does the growth in these two accounts compare? Explain your thinking.

6. Match the equivalent expressions.

a. $3^9 \cdot 3^2$	....	3^4
b. $(3^3)^2$	....	3^{11}
c. $3^4 \cdot 3$	....	3^5
d. $\frac{3 \cdot 3 \cdot 3}{3}$	....	3^2
e. 9^2	....	3^6
f. $\frac{3^2}{3^2}$	....	3^0

SUB-UNIT

2 A New Kind of Relationship

Narrative Connections

How did an enslaved man save the city of Boston?

In 1721, the residents of Boston were scared and angry. Smallpox was ripping through the city, killing hundreds of Bostonians. But in the end, thousands of lives were saved by a man who was knowledgeable in the practices of inoculation and also enslaved by an influential minister.

Onesimus, as he had been named, had educated the minister Cotton Mather and others on the process of "variolation," in which pus from someone already infected by smallpox was applied to the open wounds of a healthy patient. By giving the patient a small exposure to the disease, the patient's immune system would create antibodies to protect against a more lethal case of it. This practice was common in Sub-Saharan Africa and parts of Asia. But in the New World, Onesimus' variolation was feared as being too dangerous. It was indeed dangerous, particularly because those who had been inoculated were still contagious, and could further spread the disease throughout the city.

In spite of death threats and mob violence, Dr. Zabdiel Boylston treated 286 people using Oneismus' technique. By the end of the outbreak in 1722, 98% of those treated survived. Onesimus' efforts had dramatically slowed the spread of disease. (The next major outbreak would not occur until 1752.)

In 1980, the World Health Organization declared that smallpox had been completely eradicated, largely due to vaccination efforts. It is the only infectious disease (for humans) to date that has been eradicated. Understanding how diseases grow assist us in knowing how to combat them. Exponential relationships help us do just that.

Unit 4 | Lesson 4

Representing Exponential Growth

Let's explore exponential growth.

Warm-up Powers of Two

Rewrite each expression as a power of 2.

a) $2^3 \cdot 2^4$

b) $2^5 \cdot 2$

c) $2^{10} \div 2^7$

d) $2^9 \div 2$

Name: .... Date: .... Period: ....

Activity 1 Powers of Zero, Revisited

1. Complete the table.

x	3^x
4	81
3	27
2	
1	
0	

2. Use the product rule and quotient rule for exponents to determine the missing values in the given equations:

a $9^? \cdot 9^7 = 9^7$

b $\dfrac{9^{12}}{9^?} = 9^{12}$

3. What is the value of 5^0? What is the value of 2^0?

Are you ready for more?

The associative property tells you that $(2 + 3) + 5 = 2 + (3 + 5)$ and $(2 \cdot 3) \cdot 5 = 2 \cdot (3 \cdot 5)$. The grouping of the parentheses does not affect the value of the expression.

1. Is this true for exponents? In other words, does $2^{(3^5)}$ equal $(2^3)^5$? If not, which is greater?

2. Which of the two would you choose as the meaning of the expression 2^{3^5} (written without parentheses)?

Activity 2 Multiplying Microbes

1. In a biology lab, 500 bacteria reproduce by splitting. Every hour, each bacterium splits into two bacteria.

   a. Complete the table by writing an expression for the number of bacteria after each hour.

Number of hours	Number of bacteria
0	500
1	
2	
3	
6	
t	

   b. Write an equation relating the number of bacteria b to the number of hours t.

   c. Use your equation to calculate b when $t = 0$. What does this value of b represent in this scenario?

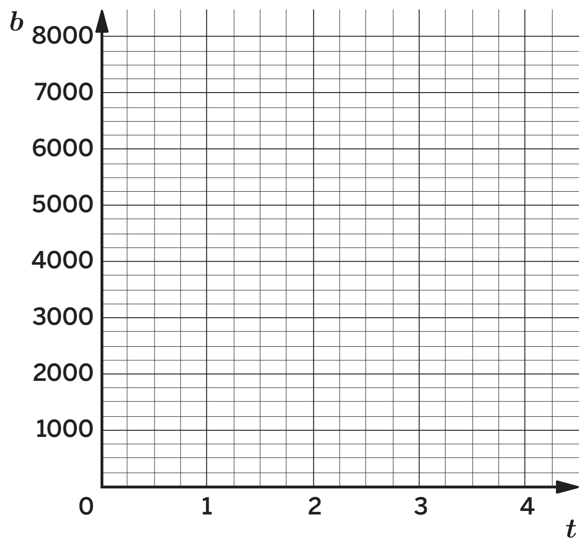
2. In another biology lab, a population of single-celled parasites is studied. An equation for the number of parasites p after t hours is $p = 100 \cdot 3^t$. Explain what the values 100 and 3 represent in this scenario.

Name: ____________ Date: ____________ Period: ____________

Activity 3 Graphing the Microbes

Refer to Activity 2 to complete these problems about the number of bacteria b and the number of single-celled parasites p.

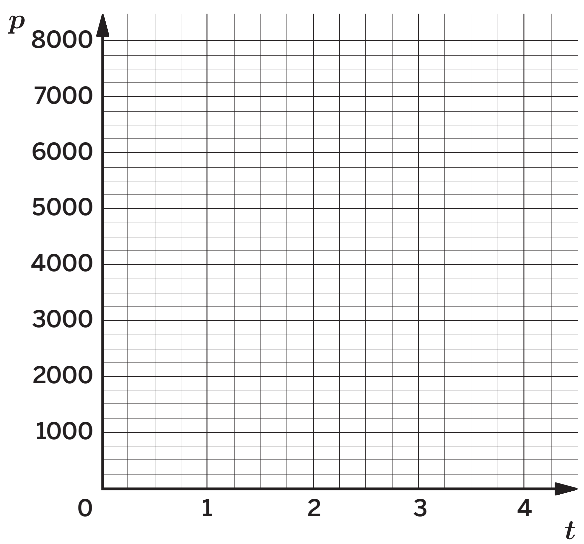
1. Graph (t, b) when t is $0, 1, 2, 3,$ and $4.$



a Rewrite the equation that relates the number of bacteria b to the number of hours t.

b On the graph of b, where can you see each number that appears in the equation?

2. Graph (t, p) when t is $0, 1, 2, 3,$ and $4.$



a Rewrite the equation that relates the number of parasites p to the number of hours t.

b On the graph of p, where can you see each number that appears in the equation?

STOP

Summary

In today's lesson . . .

You explored tables and graphs of exponential relationships. ***Exponential growth*** is a change characterized by the repeated multiplication of a common factor. The common factor in an exponential relationship is called a ***growth factor***. The general form of an exponential equation is $y = a \bullet b^x$, where a is the initial value and b is the *growth factor*. Every time x increases by 1, y is multiplied by a factor of b. The initial value a occurs when $x = 0$, and you can check this by substituting 0 for x.
$y = a \bullet b^0 = a \bullet 1 = a$

Reflect:

Name: ______________ Date: ______ Period: ______

Practice

1. Which expression is equal to $4^0 \cdot 4^2$?

A. 0

B. 1

C. 16

D. 64

2. Select *all* the expressions that are equivalent to 3^8.

A. 8^3

B. $\frac{3^{10}}{3^2}$

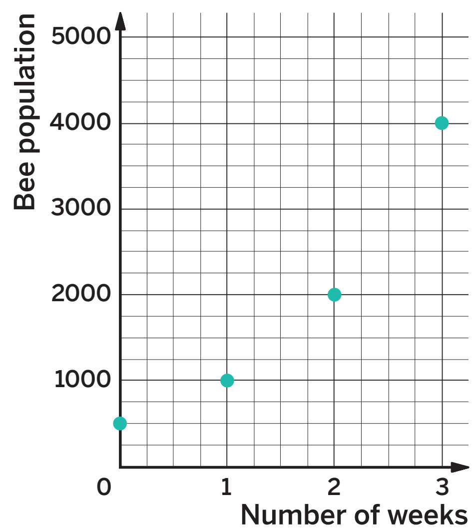
C. $3 \cdot 8$

D. $(3^4)^2$

E. $(3 \cdot 3)^4$

F. $\frac{1}{3^{-8}}$

3. A bee population is measured each week and the results are plotted on the graph.



a What is the bee population when it is first measured?

b What equation represents the bee population b for w weeks after it is first measured?

c Is the bee population growing by the same factor each week? Explain your thinking.

4. A bond is initially bought for $250. It doubles in value every decade.

a Complete the table.

Decades since bond is bought	0	1	2	3
Dollar value of bond ($)				

b How many decades will it take until the bond is worth *more* than $10,000?

c Write an equation relating the value of the bond v to the number of decades d since the bond was bought.

Practice

Name: .... Date: .... Period: ....

5. A sea turtle population p is modeled by the equation $p = 400 \cdot \left(\frac{5}{4}\right)^y$, where y is the number of years since the population is first measured.

a How many turtles are in the population when it is first measured? Where do you see this in the equation?

b What is the growth factor? What does it mean in context?

c When will the turtle population reach 700? Explain your thinking.

6. Determine whether each table represents a linear relationship. If so, write an equation of the line that passes through the given points.

a

x	-3	-2	2	5
y	2.5	1	-5	-9.5

b

x	-3	-2	0	2
y	-0.5	1	4	16

c

x	-5	-2	2	8
y	1.2	3	5.4	9

Name: ______ Date: ______ Period: ______

Unit 4 | Lesson 5

Understanding Decay

Let's look at exponential decay.

Warm-up Notice and Wonder

Study the tables. What do you notice? What do you wonder?

Table A

x	0	1	2	3	4
y	2	$3\frac{1}{2}$	5	$6\frac{1}{2}$	8

Table B

x	0	1	2	3	4
y	2	3	$\frac{9}{2}$	$\frac{27}{4}$	$\frac{81}{8}$

1. I notice . . .

2. I wonder . . .

Log in to Amplify Math to complete this lesson online.

Activity 1 What Is Left?

1. Diego has $\$100$ and will spend a fourth of it. He needs to determine how much money he will have left. Complete the following table by explaining what he did in each step.

Statement	Reason
$100 - \frac{1}{4} \cdot 100$	
$100\left(1 - \frac{1}{4}\right)$	
$100 \cdot \frac{3}{4}$	
$\frac{3}{4} \cdot 100$	
75	

2. Mai makes $\$1{,}800$ per month, but spends $\frac{1}{3}$ of that amount for rent. What two numbers could be multiplied to determine how much money she has after paying rent?

3. Write a multiplication expression that is equivalent to x reduced by $\frac{1}{8}$ of x.

Name: ________ Date: ________ Period: ________

Activity 2 Value of a Vehicle

A new car costs $18,000. Each year after a new car is purchased, it loses $\frac{1}{3}$ of its value.

1. A buyer worries that the car will be worth nothing in three years. Do you agree? Explain your thinking.

2. Complete the table by writing an expression to show how to determine the value of the car for each year listed.

Year	Value of car ($)
0	$18{,}000$
1	$18{,}000 \bullet \frac{2}{3}$
2	$18{,}000 \bullet \frac{2}{3} \bullet \frac{2}{3}$ or $18{,}000 \bullet \left(\frac{2}{3}\right)^2$
3	
6	
t	

3. Write an equation relating the value of the car in dollars v to the number of years t.

4. Use your equation to determine the value of v when t is 0. What does this value of v represent in this scenario?

5. A different car loses value at a different rate. The value of this different car in dollars d after t years can be represented by the equation $d = 10000 \bullet \left(\frac{4}{5}\right)^t$. Explain what the values $10{,}000$ and $\frac{4}{5}$ represent in this scenario.

Activity 2 Value of a Vehicle (continued)

Are you ready for more?

Start with an equilateral triangle whose area is 1 square unit, divide it into 4 congruent pieces, and remove the middle piece. Then, repeat this process for each of the remaining pieces. The following figure shows the first several steps of this construction.

1. What fraction of the remaining area is removed at each step?

2. What is the total remaining area after the nth step?

3. Use a calculator to determine the total remaining area after 50 steps.

Name: ________ Date: ________ Period: ________

Activity 3 Exponential Success of the Polio Vaccine

Polio is a highly infectious disease for which there is no cure, but which has been nearly eradicated by vaccination. Incidence of polio peaked in the U.S. in 1952, when there were 57,879 reported cases. In 1953, the U.S. Public Health Service began a trial vaccination program with more than 1.8 million school children. In 1955, it launched a nationwide vaccination program. The World Health Assembly began a global vaccination program in 1988, and by 2017, the number of cases reported globally had been reduced by 99.99%. In 2020, about 1,200 cases were reported globally.

Examine the table showing the reported number of polio cases in the U.S. between 1952 and 1960.

Year	Number of reported Polio cases in the U.S.
1952	57,879
1953	35,592
1955	28,985
1956	15,140
1957	5,485
1960	3,190

1. Do the number of reported cases appear to decrease exponentially? Explain your thinking.

2. Elena observes an exponential decay pattern and uses graphing technology to find a model equation. She begins with 1952 as year 0 and finds a decay factor of approximately 0.695. How many cases does her model, $n = 57879 \cdot (0.695)^t$, predict for the year 1960?

3. How does the predicted value from Elena's model compare to the actual number of cases reported?

4. Does this support your original observation of the data? Explain your thinking.

STOP

Summary

In today's lesson . . .

You looked at ***exponential decay***, or quantities that decrease by the same factor repeatedly. Previously, you studied quantities that increased by the same factor repeatedly. While this is still called a *growth factor*, it is also called a ***decay factor*** when the factor is between 0 and 1 — because repeated multiplication by a positive factor less than 1 results in decreasing values.

Reflect:

Practice

Name: ______ Date: ______ Period: ______

1. A new bicycle sells for $\$300$. It is on sale for $\frac{1}{4}$ off the regular price. Select *all* the expressions that represent the sale price of the bicycle in dollars.

   A. $300 \cdot \frac{1}{4}$

   B. $300 \cdot \frac{3}{4}$

   C. $300 \cdot \left(1 - \frac{1}{4}\right)$

   D. $300 - \frac{1}{4}$

   E. $300 - \frac{1}{4} \cdot 300$

2. A computer costs $\$800$. It loses $\frac{1}{4}$ of its value every year after it is purchased.

   a. Complete the table to show the value of the computer at the listed times.

   b. Write an equation representing the value v of the computer, t years after it is purchased.

Time (years)	Value of computer ($)
0	
1	
2	
3	
t	

   c. Use your equation to determine the value of v when t is 5. What does this value of v mean?

3. A piece of paper is folded into thirds multiple times. The area A, in square inches, of the piece of paper after n folds is $A = 90 \cdot \left(\frac{1}{3}\right)^n$.

   a. What is the value of A when $n = 0$? What does this represent in this scenario?

   b. How many folds are needed so that the area is less than 1 in^2?

   c. The area B, in square inches, of another piece of paper after n folds, is given by the equation $B = 100 \cdot \left(\frac{1}{2}\right)^n$. What do the values 100 and $\frac{1}{2}$ mean in this situation?

Practice

Name: ________ Date: ________ Period: ________

4. At the beginning of April, a colony of ants has a population of $5{,}000$.

a The colony decreases by $\frac{1}{5}$ during April. Write an expression for the ant population at the end of April.

b During May, the colony decreases again by $\frac{1}{5}$ of its size. Write an expression for the population of the ant colony at the end of May.

c The colony continues to decrease by $\frac{1}{5}$ of its size each month. Write an expression for the ant population after 6 months.

5. An odometer is the part of a car's dashboard that shows the number of miles a car has traveled in its lifetime. Before a road trip, a car odometer reads $15{,}000$ miles. During the trip, the car travels 65 miles per hour.

a Complete the table.

b What do you notice about the differences in the odometer readings each hour?

c If the odometer reads n miles at a particular hour, what will it read one hour later?

Duration of trip (hours)	Odometer reading (miles)
0	
1	
2	
3	
4	
5	

6. Lin evaluated the expression $-5x + 3^x$ for $x = 2$. Determine the mistake she made in the work shown and then evaluate the expression correctly.

Lin's work:

$$\begin{aligned} & -5(2) + 3^2 \\ = & -10 + 3^2 \\ = & (-7)^2 \\ = & \ 49 \end{aligned}$$

Name: ________________________ Date: __________ Period: __________

Unit 4 | Lesson 6

Representing Exponential Decay

Let's think about how to represent exponential decay.

Warm-up Two Tables

Use the patterns you notice to complete the tables. Explain your thinking.

Pattern A

x	y
0	2.5
1	10
2	17.5
3	25
4	
25	

Pattern B

x	y
0	2.5
1	10
2	40
3	160
4	
25	

Log in to Amplify Math to complete this lesson online.

Activity 1 The Algae Bloom

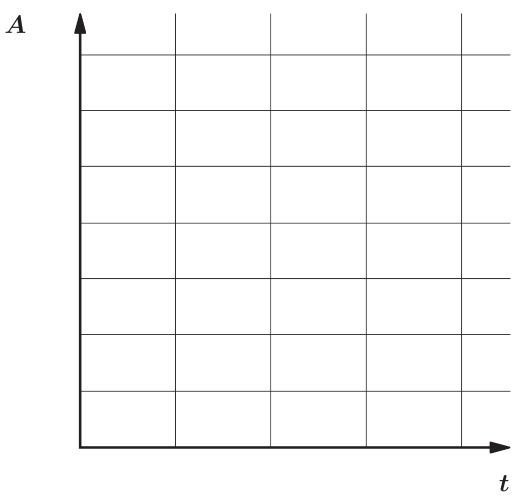
An algae bloom is a rapid increase in the population of algae in a freshwater lake. Harmful algae blooms (HABs) produce biotoxins in lakes that can cause illness in humans and animals that are directly exposed to them or that eat seafood contaminated by HAB toxins. To control a bloom in a lake, scientists can introduce treatment chemicals.

Three Reads:
You will read this introduction three times to help make sense of the scenario. Your teacher will tell you what to look for during each read.

After treatment began at a certain lake, the area covered by algae A, in square yards, is given by the equation $A = 240 \cdot \left(\frac{1}{3}\right)^t$. The time t is measured in weeks.

1. In the equation, what does the value 240 tell you about the algae? What does the value $\frac{1}{3}$ tell you?

2. Create a graph that represents $A = 240 \cdot \left(\frac{1}{3}\right)^t$ when t is 0, 1, 2, 3, and 4. Think carefully about how you choose the scales for the axes. (Create a table of values if that is helpful.)



3. About how many square yards will the algae cover after 2.5 weeks? Explain your thinking.

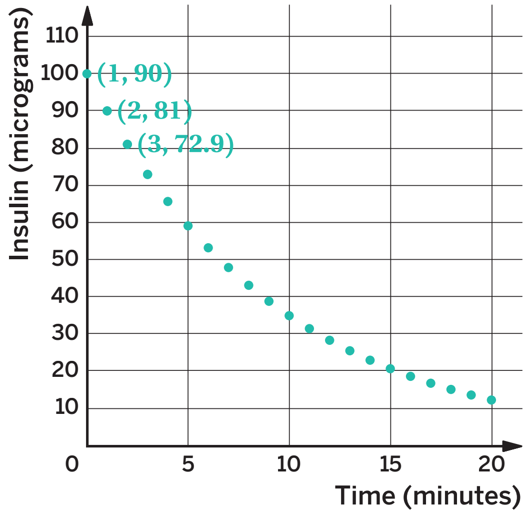
Are you ready for more?

To keep the algae from spreading further, the scientists estimate that the area must be less than 1 ft^2. For how many weeks should they run the treatment to achieve this?

Name: .... Date: .... Period: ....

Activity 2 Insulin in the Body

A patient who is diabetic receives 100 micrograms of insulin. The graph shows the amount of insulin remaining in his bloodstream over time.



1. Researchers believe the amount of insulin in a patient's body changes exponentially. How can you check if the graph supports the researchers' claim?

2. How much insulin was metabolized in the first minute? What fraction of the original insulin is that?

3. How much insulin was metabolized in the second minute? What fraction is that of the amount one minute earlier?

4. What fraction of insulin *remains* in the bloodstream after each minute? Justify your answer.

5. Complete the table, showing the predicted amount of insulin 4 and 5 minutes after injection.

Time after injection (minutes)	0	1	2	3	4	5
Insulin in the bloodstream (micrograms)	100	90	81	72.9		

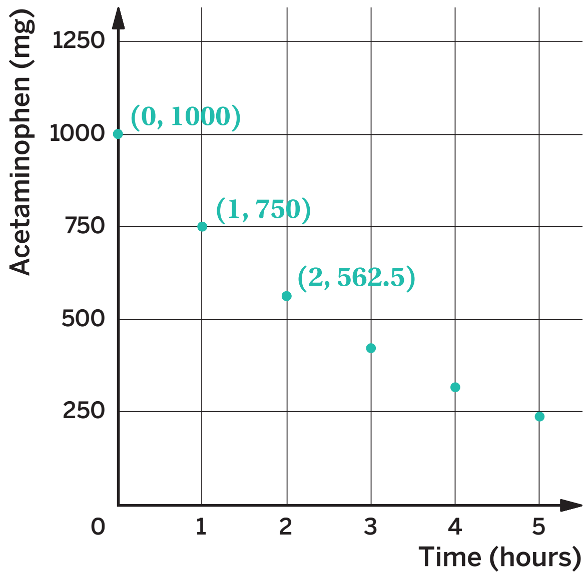
6. Describe how you would determine the amount of insulin remaining in his bloodstream after 10 minutes. After m minutes?

STOP

Summary

In today's lesson . . .

You examined several graphs that represented exponential decay.
For example, the following graph shows the total amount of acetaminophen in an adult's body at different times after they take a normal dose.



The initial value is represented by the point $(0, 1000)$. This means that the initial amount of acetaminophen in an adult's body, (or normal adult dose), is $1{,}000$ mg.

You can use the graph to determine the fraction of acetaminophen that remains in the body from one hour to the next — the common factor, or the *decay factor.* As each hour passes, the amount of acetaminophen that remains in the body is multiplied by a decay factor of $\frac{3}{4}$.

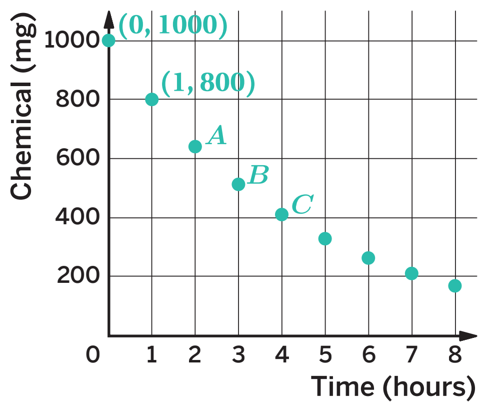
If y is the number of milligrams of acetaminophen in an adult's body x hours after they take a normal dose, then this scenario is modeled by the equation $y = 1000 \bullet \left(\frac{3}{4}\right)^x$. It still has the same general form of an exponential equation $y = a \bullet b^x$. In the case of exponential decay, however, the common factor b is between 0 and 1.

Reflect:

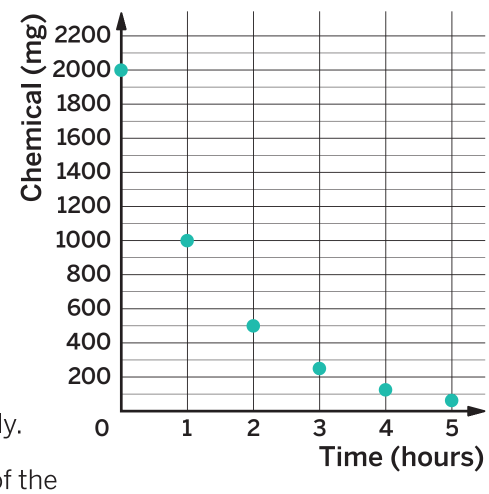
Name: ________________ Date: ________ Period: ________

Practice

1. The graph shows the amount of a chemical in a water sample. It is decreasing exponentially. Find the coordinates of the points labeled A, B, and C. Explain your thinking.



2. The graph shows the amount of a chemical in another water sample at different times after it was first measured. Select *all* statements that are true.



   A. The amount of the chemical in the water sample is decreasing exponentially.

   B. The amount of the chemical in the water sample is not decreasing exponentially.

   C. It is not possible to tell for certain whether the amount of the chemical is decreasing exponentially.

   D. When it was first measured, there were $2{,}000$ mg of the chemical in the water sample.

   E. After 4 hours, there were 100 mg of the chemical in the water.

3. The number of people who have read a new book is 300 at the beginning of January and doubles each month.

   a. Complete the table.

Number of months since January	0	1	2	3	4
Number of people who have read the book					

   b. What do you notice about the difference in the number of people who have read the book from month to month?

Practice

Name: .... Date: .... Period: ....

c What do you notice about the factor by which the number of people changes each month?

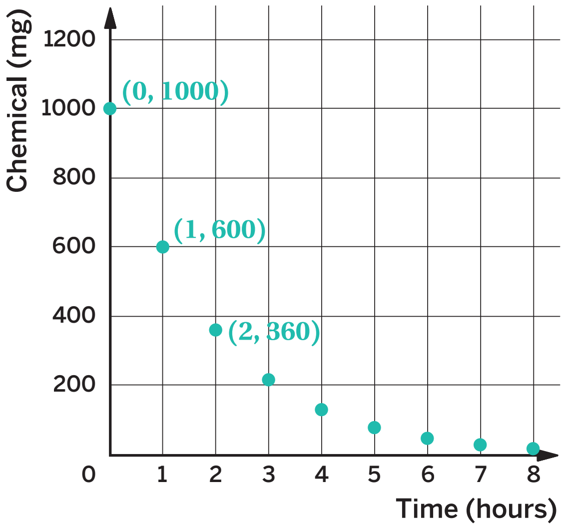
d If n people read the book in one month, how many people will read the book the following month?

4. Solve each system of equations. Show your thinking.

a $\begin{cases} x + y = 2 \\ -3x - y = 5 \end{cases}$

b $\begin{cases} \frac{1}{2}x + 2y = -13 \\ x - 4y = 8 \end{cases}$

5. The function $g(h)$ represents the amount of a chemical in a patient's body every hour since the levels were first checked. Select *all* of the statements that are true about the function.



A. $g(360) = 2$

B. $g(0)$ represents the initial value or the initial amount of milligrams of the chemical in the patient's body.

C. $g(4) > g(5)$

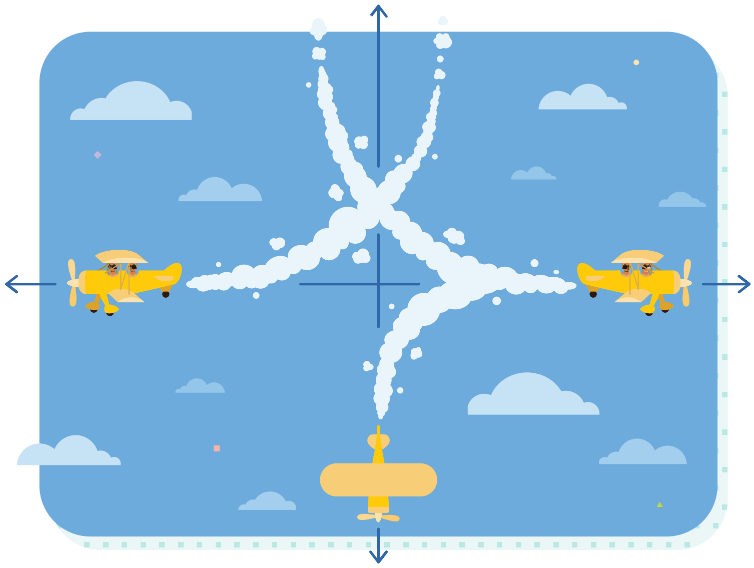
D. After 1 hour, the patient had 600 mg of the chemical left in their system.

E. The growth rate is $\frac{5}{3}$.

Name: .... Date: .... Period: ....

Unit 4 | Lesson 7

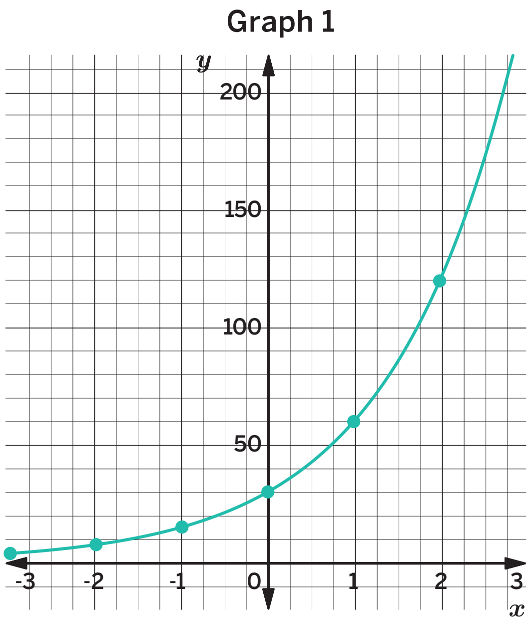
Exploring Parameter Changes of Exponentials

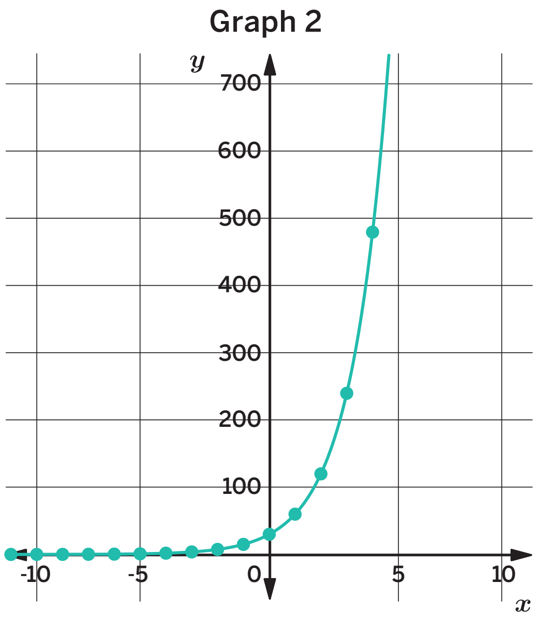


Let's examine how changing an exponential equation changes its graph.

Warm-up Would You Rather?

Consider the following graphs.





Suppose your teacher plans to give you a quiz on exponential graphs. You will need to identify the initial value and the growth factor, write an equation for the graph, and state what their coordinates represent. Would you rather complete the quiz using Graph 1 or Graph 2? Explain your thinking.

Log in to Amplify Math to complete this lesson online.

Activity 1 Changing b

Let's see what happens when we replace b with its reciprocal in the equation $y = a \cdot b^x$.

1. Consider the equations $y = 3 \cdot 2^x$ and $y = 20 \cdot \left(\frac{1}{5}\right)^x$. Complete the following for both equations.

a Identify the values for a and b.

b Does this equation represent exponential growth or decay? Explain your thinking.

c Determine the reciprocal of b. Then rewrite the original equation, replacing b with its reciprocal.

d Use a graphing tool to graph the equation and its new equation you wrote in part c. Compare the graphs. Describe how the reciprocal of b changes the graph.

2. Make a conjecture about how replacing b with its reciprocal changes the graph of an exponential function.

Name: ________ Date: ________ Period: ________

Activity 2 Negating a

Let's see what happens when we negate a in the equation $y = a \cdot b^x$.

1. Consider the equations $y = 5 \cdot \left(\frac{5}{4}\right)^x$ and $y = -4 \cdot \left(\frac{2}{3}\right)^x$. Complete the following for both equations.

   a. Identify the values for a and b.

   b. Do these equations represent exponential growth or decay? Explain your thinking

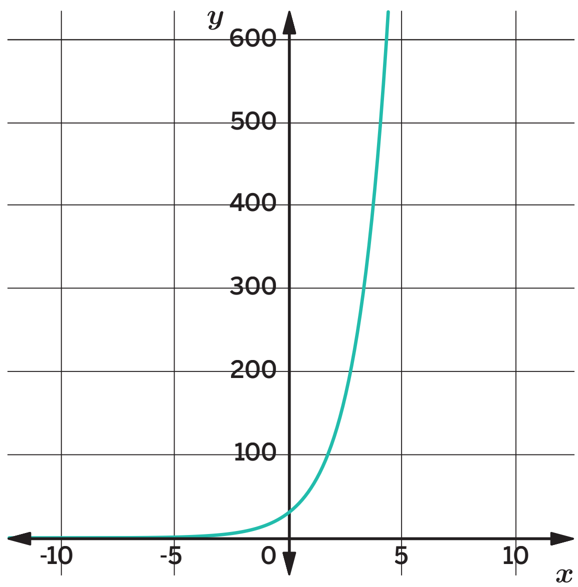
   c. Determine the opposite of a. Then rewrite the given equations, replacing a with its opposite.

   d. Use a graphing tool to graph the equation and its new equation you wrote in part c. Compare the graphs. Describe how negating a changes the graph.

2. Make a conjecture about how negating a changes the graph of an exponential function.

Activity 3 Adjusting the Axes

Use the graph of the exponential equation to complete these problems.



1. What is the value of y when $x = 0$? Can you say for sure?

2. What is the value of y when $x = 1$? Can you say for sure?

3. What is the value of y when $x = 3$? Can you say for sure?

4. If $(5, 960)$, $(6, 1920)$, and $(7, 3840)$ are points that satisfy this equation, what is the growth factor? Explain your thinking.

5. How would you adjust your axes to view these coordinates?

STOP

Name: ______ Date: ______ Period: ______

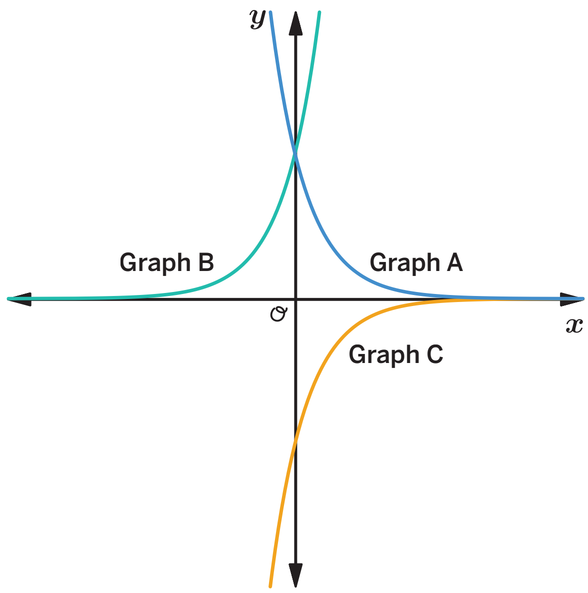
Summary

In today's lesson . . .

You examined how changing the values of a and b in an equation of the form $y = a \cdot b^x$ affects its graph.

When you replace the value of the growth factor b with its reciprocal, it reflects the graph across the y-axis. For example, Graph B is a reflection, across the y-axis, of Graph A.

When you negate the initial value a, it reflects the graph across the x-axis. For example, Graph C is a reflection, across the x-axis, of Graph A.



Reflect:

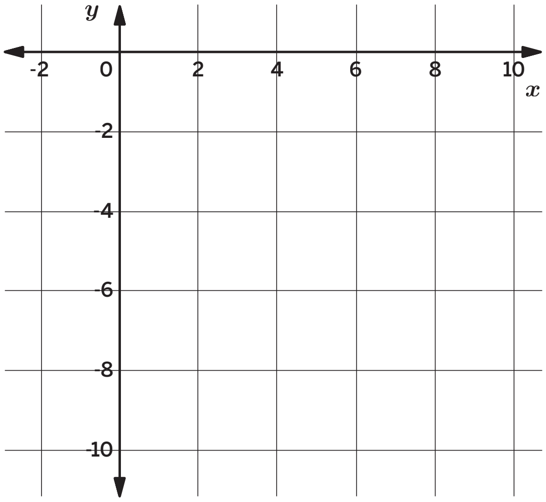
Name: ______________________ Date: ____________ Period: ____________

1. Bard and Andre take the reciprocal of b in the equation, $y = 4 \cdot \left(\frac{1}{3}\right)^x$. Bard states, "The new equation represents exponential decay because $\frac{1}{4}$ is less than 1." Andre disagrees. Do you agree with Bard or Andre? Explain your thinking.

2. Lin studies the equation $y = 5 \cdot 2^x$. She takes the additive inverse of a and the reciprocal of b. She graphs the new equation.

a What is Lin's new equation?

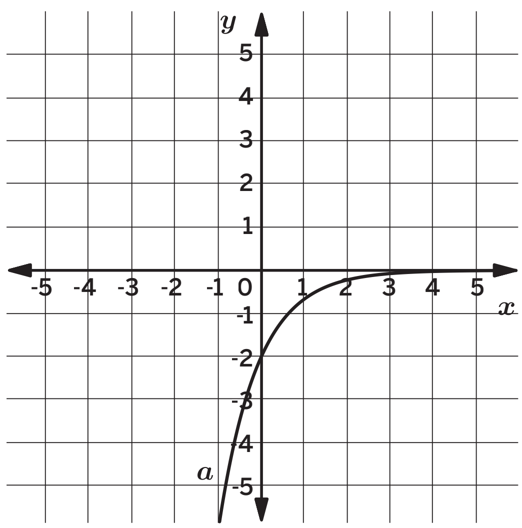
b Graph Lin's new equation.



3. The graph of $y = -2 \cdot \left(\frac{1}{3}\right)^x$ is shown. Clare takes the additive inverse of a.

a What is Clare's new equation?

b Graph her new equation on the graph.



Name: ____________ Date: ____________ Period: ____________

4. A popular cell phone costs \$950. During a promotion it sells for $\frac{1}{5}$ of its regular price. Which expression best represents the sale price of the cell phone in dollars?

A. $950 \cdot \frac{1}{5}$

B. $950 \cdot \left(\frac{1}{5}\right)^x$

C. $950 - \frac{1}{5}$

D. $950 \cdot \left(1 - \frac{1}{5}\right)$

5. Elena collects data to investigate the relationship between the number of bananas she buys at the store x, and the total cost of the bananas y. Which value for the correlation coefficient is most likely to match a line of best fit of the form $y = mx + b$ for this situation?

A. -0.9

B. -0.4

C. 0.4

D. 0.9

6. Determine the decimal and percent equivalent of each fraction.

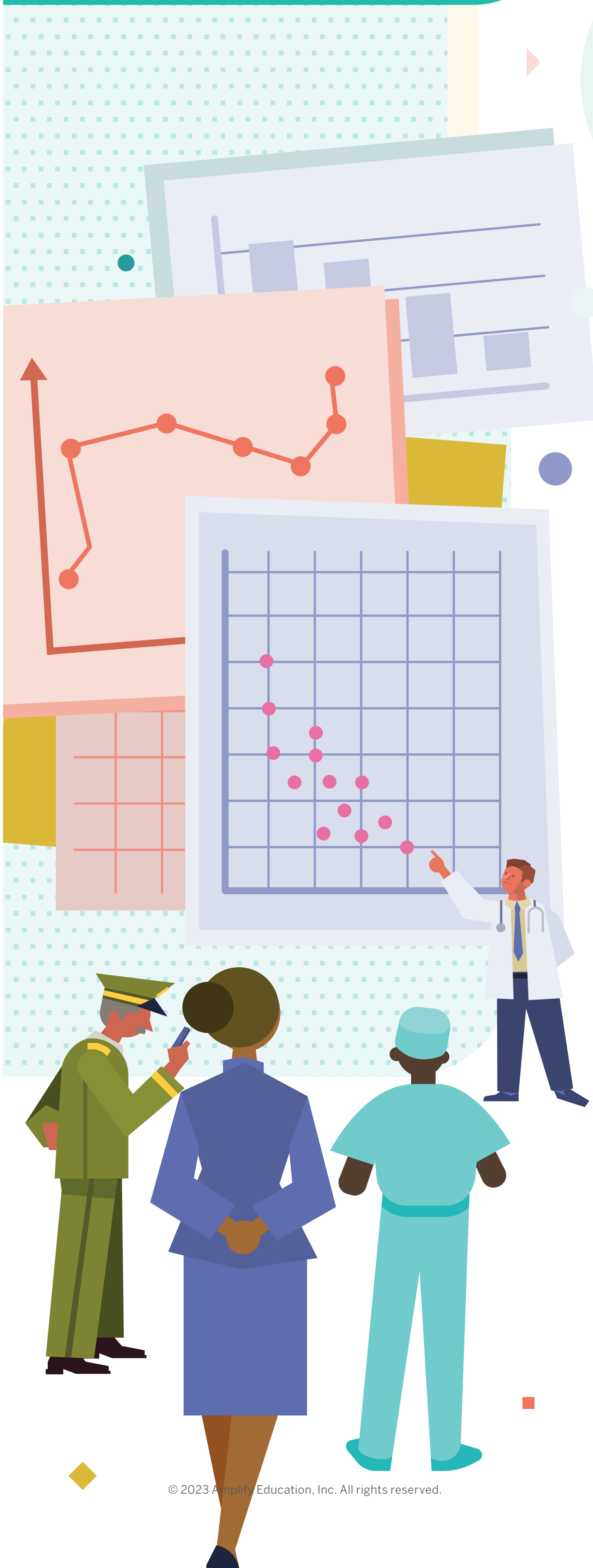
Fraction	Decimal	Percent
$\frac{1}{5}$		
$\frac{1}{10}$		
$\frac{1}{20}$		

My Notes:

SUB-UNIT

3 Exponential Functions

Narrative Connections



What does growing or shrinking look like on a graph?

Math has given us tools that help us understand the meaning behind numbers, from the staggeringly large to the painstakingly minuscule. Representations like tables and graphs let us see things in numbers that we might not fully otherwise grasp.

Think back to Lesson 1, when you simulated the spread of an infection, starting with just one infected person. Could you have imagined how quickly it spread to so many other people? By using a graph, we can see with our own eyes the magnitude of an epidemic. And once we see it, we can be smarter about how we fix it.

With charts and tables, we can see more of the story — from the first outbreak, to the waves of infection, to when a vaccine is developed and introduced. We can see the people who have developed natural immunity, and the effects of treatments, vaccinations, and quarantines. Armed with this information, government officials, scientists, and doctors can make more informed decisions that can save lives.

Unit 4 | Lesson 8

Analyzing Graphs

Let's explore exponential growth and decay by comparing situations where quantities change exponentially.

Warm-up Fractions and Decimals

In the table, find as many patterns as you can.

Fraction	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{32}$
Decimal	0.5	0.25	0.125		

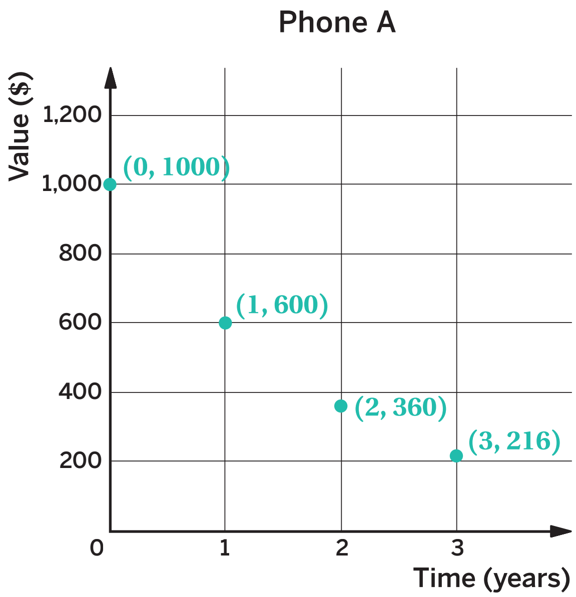
Use one or more patterns to help you complete the table. Explain your thinking.

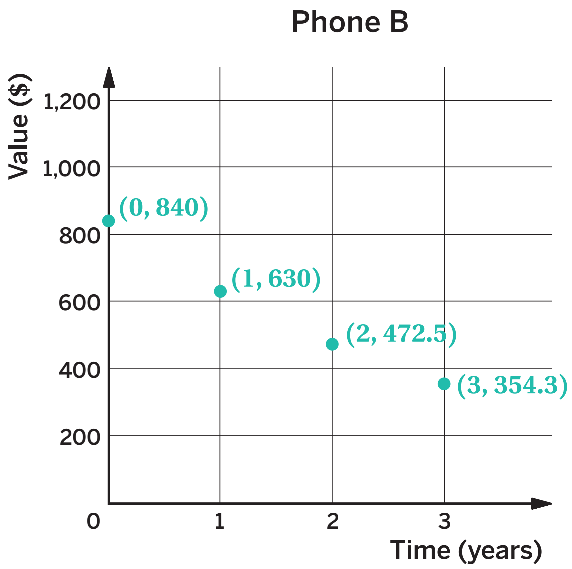
Log in to Amplify Math to complete this lesson online.

Name: ____________ Date: ____________ Period: ____________

Activity 1 Falling and Falling

The value of some cell phones changes exponentially after the initial release. Here are graphs showing the depreciation of Phone A and Phone B one, two, and three years after they were released.





1. Which phone was more expensive when it was first released?

2. How does the value of each phone change with every passing year?

3. Which one loses its value faster? Explain your thinking.

4. If the phones continue depreciating by the same factors each year, what will be the value of each phone 4 years after its initial release?

Activity 1 Falling and Falling (continued)

5. For each cell phone, write an equation that relates the value of the phone in dollars to the years t since release. Use v for the value of Phone A and w for the value of Phone B.

Are you ready for more?

It is not always clear how to best model data you are given. In this case, you were told the values of the cell phones were changing exponentially. Suppose you were only given the initial values of the cell phones and their values after each of the first three years.

1. Assuming the values decrease linearly, use technology to compute the line of best fit for each cell phone. Round any values to the nearest whole number. Using this model, would values decrease by a common difference or a common factor?

2. Explain why, in this situation, an exponential model might be more appropriate than the linear model you just created. How are the cell phone values decreasing?

Name: ______________________ Date: ____________ Period: ____________

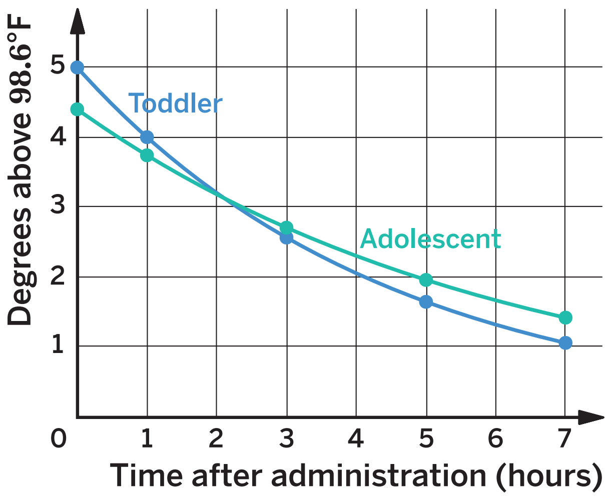
Activity 2 Card Sort: Matching Descriptions to Graphs

You will be given a set of cards containing descriptions of scenarios and graphs. Match each scenario with a graph that represents it. Record your matches and be prepared to explain your thinking. Record any observations in the table.

Card	Corresponding graph

Activity 3 Fever Reduction

A caregiver administers two different brands of fever reducers to their toddler and adolescent. The graphs show the fevers, measured in degrees Fahrenheit above a healthy temperature of $98.6°F$, of each child over the course of 7 hours. The toddler's fever is reduced by $\frac{1}{5}$ each hour, while the adolescent's fever is reduced by $\frac{3}{20}$ each hour.



1. Which child has the higher temperature at the start of administration of the fever reducer?

2. Which child's fever decreases faster?

3. Using d degrees above a healthy temperature in t hours after administration of the fever reducer, write equations to model the toddler's and the adolescent's fevers.

STOP

Name: .... Date: .... Period: ....

Summary

In today's lesson . . .

You continued examining exponential decay through real-world situations, graphs, and equations.

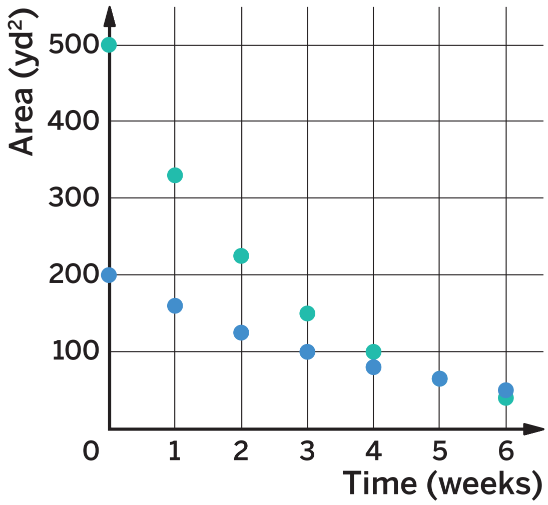
Growth factors greater than 1 describe exponential growth, while growth factors between 0 and 1 (in which case they are commonly called decay factors) describe exponential decay.

Reflect:

Practice

Name: .... Date: .... Period: ....

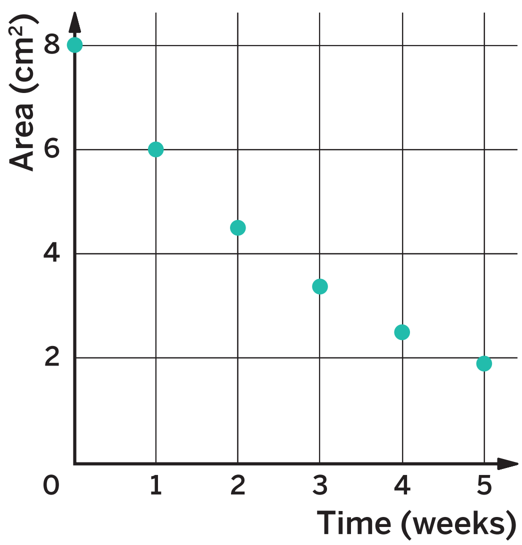
1. The two graphs show models for the area covered by two algae blooms, in square yards, w weeks after different chemicals were applied.



a Which bloom covered a greater area when the chemicals were first applied? Explain your thinking.

b Which bloom's population is decreasing faster? Explain your thinking.

2. A medicine is applied to a burn on a patient's arm. The area of the burn in square centimeters decreases exponentially and is shown in the graph.



a What fraction of the burn area remains each week?

b Write an equation representing the area A of the burn, after t weeks?

c What is the area of the burn after 7 weeks? Round to three decimal places.

Name: ______________________ Date: ____________ Period: ____________

Practice

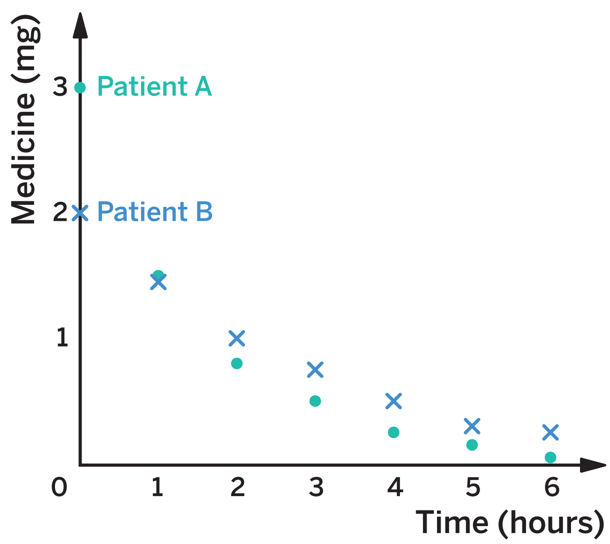
3. The following problems are about folding pieces of paper.

a. The area of a sheet of paper is 100 in^2. Write an equation that gives the area A, in square inches, of a sheet of paper that is folded in half n times.

b. The area of another sheet of paper is 200 in^2. Write an equation that gives the area B, in square inches, of a sheet of paper that is folded into thirds n times.

c. Are the areas of the two sheets of paper ever the same after each being folded n times? Show or explain your thinking.

4. The graphs show the amount of medicine in two patients after receiving injections. The dots show the medicine in Patient A and the x's show the medicine in Patient B. An equation that gives the amount of medicine m, in milligrams, in Patient A after h hours is $m = 300 \cdot \left(\frac{1}{2}\right)^h$. What is a possible equation for the amount of medicine in Patient B?



A. $m = 500 \cdot \left(\frac{3}{10}\right)^h$

B. $m = 500 \cdot \left(\frac{7}{10}\right)^h$

C. $m = 200 \cdot \left(\frac{3}{10}\right)^h$

D. $m = 200 \cdot \left(\frac{7}{10}\right)^h$

5. Select *all* expressions that are equivalent to 3^{-8}.

A. $(3^{-2})^{-4}$

B. $\frac{3^4}{3^{12}}$

C. $(3^{-9})^1$

D. $(3^4)^{-2}$

E. $3^6 \cdot 3^{-2}$

F. $3^{-10} \cdot 3^2$

Unit 4 | Lesson 9

Using Negative Exponents

Let's study exponential graphs and equations more closely.

Warm-up Which Expressions Are Equivalent?

Sort the following expressions into groups of equivalent expressions.

$2^4 \cdot 2^0$	$2^3 \cdot 2^{-3}$	$2^2 \cdot 2^{-4}$
$\frac{1}{2^2}$	2^4	$\frac{2^6}{2^2}$
2^{-2}	1	2^0

Log in to Amplify Math to complete this lesson online.

Name: ________ Date: ________ Period: ________

Activity 1 Coral in the Sea

Coral reefs are incredible, colorful, underwater communities with hundreds of thousands of minuscule animals. Coral reefs act as a natural water filter, protect shorelines, and provide researchers with ingredients for life-saving medicines.

A marine biologist estimates that a certain structure of coral has a volume of 1,200 cm^3, and that its volume doubles each year.

Volodymyr Goinyk/Shutterstock.com

1. Write an equation of the form $y = a \cdot b^t$ representing the relationship between the time t in years since the coral was measured and the volume y of the coral in cubic centimeters.

2. In your equation, what do the variables a and b represent from this scenario?

3. Determine the volume of the coral when t is -2, -1, 0, 1, and 5.

Time, t (years)	Volume, y (cm^3)
-2	
-1	
0	
1	
5	

4. In this scenario, what does it mean when $t = -2$?

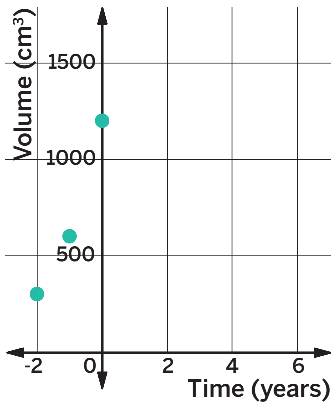
5. In which year was the volume of the coral 37.5 cm^3? Explain your thinking.

Activity 1 Coral in the Sea (continued)

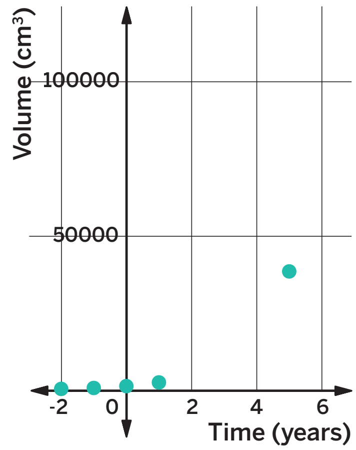
6. Mai records the volume of the coral over a 10-year span.

   a. Each of the following three graphs is correct. Which graph *best* shows the volume of the coral in the given time frame? Explain your thinking.

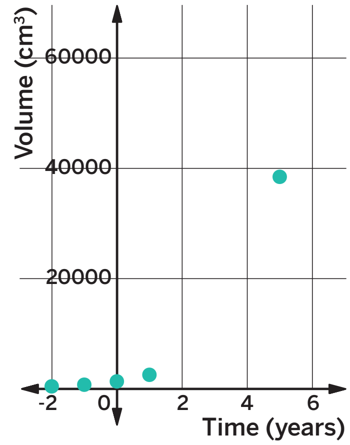
Graph A



Graph B



Graph C



   b. Explain how the volume of coral could be misinterpreted in the other graphs.

Are you ready for more?

Without evaluating them, describe each of the following quantities as *close to zero*, *close to one*, or *much larger than one*.

a. $\dfrac{1}{1-2^{-10}}$

b. $\dfrac{2^{10}}{2^{10}+1}$

c. $\dfrac{2^{-10}}{2^{10}}+1$

d. $\dfrac{1-2^{-10}}{2^{10}}$

e. $\dfrac{1+2^{10}}{2^{-10}}$

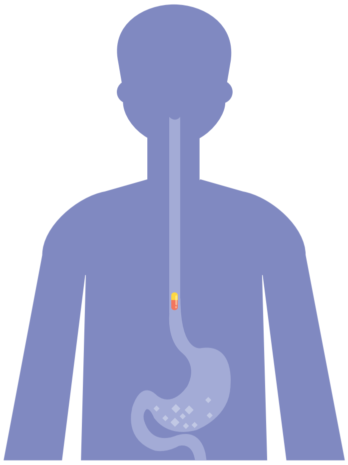
Name: .... Date: .... Period: ....

Activity 2 Measuring Medicine

Researchers like Megan Sawyer model how the human body responds to different doses of chemical compounds. In general, the amount of a compound added to the body will decrease over time.

At a hospital, a healthcare worker gives a patient some medicine at noon and measures the amount of medicine remaining in her bloodstream each hour. The patient decides to record the measurements beginning at 5 p.m. She does not know the original amount the healthcare worker gave her and she did not write down the amounts recorded at 1 p.m., 2 p.m., 3 p.m., or 4 p.m. The table shows the amounts she did record, where t is the time in hours since 5 p.m. and the amount of medicine m in her bloodstream is measured in milligrams.

Time, t (hours)	Medicine, m (mg)
0	100
1	50
2	25



1. Use the table to determine the growth factor. How much medicine was in the patient's bloodstream when she began recording the amounts at 5 p.m.?

2. Write an equation for m in terms of t.

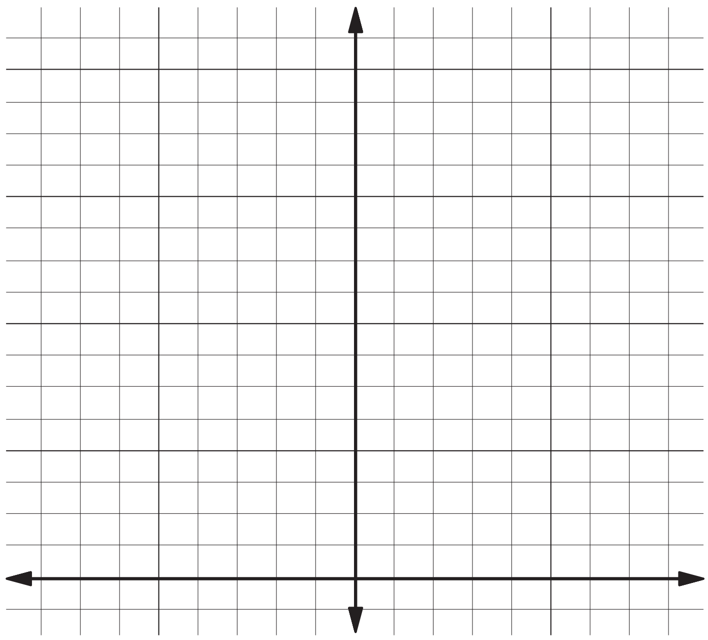
3. Determine the amount of medicine in the woman's bloodstream when t is -1 and -3. Record them in the table.

4. What do $t = 0$ and $t = -3$ represent in this context?

Activity 2 Measuring Medicine (continued)

5. The medicine was taken when t was -5. Assuming the woman did not have any of the medication in her body beforehand, how much medicine was she given at that time?

6. Plot the points whose coordinates are in the table. Make sure to label the axes and choose appropriate scales.



Featured Mathematician

Megan Sawyer

Megan Sawyer is an Associate Professor of Mathematics at Southern New Hampshire University. Their research focuses on using differential equations to model vitamin D dosing regimens, particularly in patients with chronic kidney disease. Beyond their research, Sawyer has developed new applied mathematical courses at the university, while also serving on task groups to improve campus life.

STOP

Name: ________ Date: ________ Period: ________

Summary

In today's lesson . . .

You extended your interpretation of exponential equations to include negative exponents. In particular, negative values of time t refer to times before $t = 0$ (not a negative amount of time). You can use these equations to understand what occurs before and after a certain time.

Reflect:

Practice

Name: .... Date: .... Period: ....

1. A forest fire burns for several days. The burned area, in acres, is given by the equation $y = 4800 \cdot (2)^d$, where d is the number of days since the area was first measured. First, complete the following table.

Days since first measurement, d	Acres burned since fire started, y
-5	
-3	
-2	
-1	
0	

a What does $d = -1$ mean in this context? What about $d = -3$?

b How much area had the fire burned *a week* before it measured 4,800 acres? Explain your thinking.

2. A fish population p can be represented by the equation $p = 800 \cdot \left(\frac{1}{2}\right)^t$, where t is the time in years since the beginning of 2015. What was the fish population at the beginning of 2012?

A. 100 **B.** 800 **C.** 2,400 **D.** 6,400

3. The value of a home in 2015 was \$400,000. Its value doubles each decade.

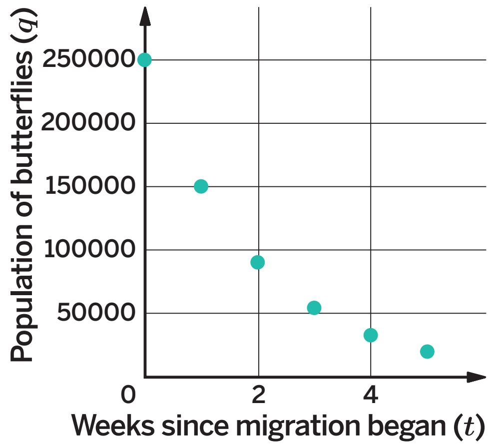
a If v is the value of the home, in dollars, write an equation for v in terms of d, the number of decades since 2015.

b What is v when $d = -1$? What does this value represent?

c What is v when $d = -3$? What does this value represent?

Name: .... Date: .... Period: ....

4. The graph shows a population q of butterflies, t weeks since their migration began.



a. What was the population at the start of the migration? Explain your thinking.

b. What was the population after one week? After two weeks?

c. Write an equation for the butterfly population q, after t weeks.

5. A book sold $600{,}000$ copies the year it was released. Each year after, the number of copies sold decreased by half.

a. Complete the table.

b. Write an equation representing the number of copies c sold, y years after the book was released.

c. Use your equation to find c when $y = 6$. What does this represent in terms of the book sales?

Years since published	Number of copies sold
0	
1	
2	
3	
y	

6. What is one way you can determine if a relation is a function?

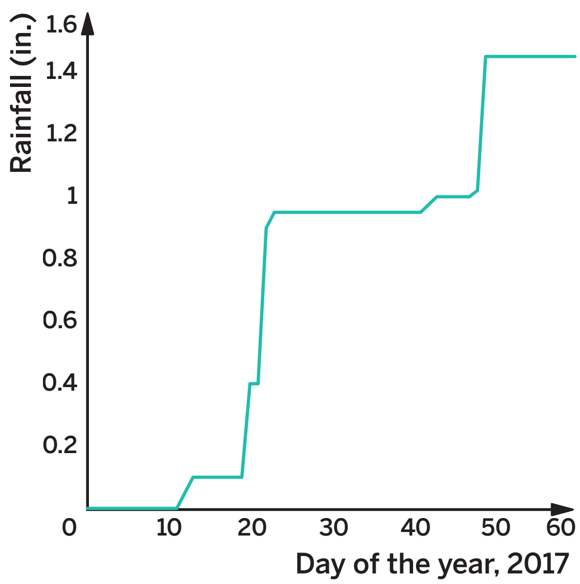
Unit 4 | Lesson 10

Exponential Situations as Functions

Let's explore exponential functions.

Warm-up Rainfall in Las Vegas

Here is a graph of the accumulated rainfall in Las Vegas, Nevada, in the first 60 days of 2017.



1. Is the accumulated amount of rainfall a function of time? Explain your thinking.

2. Is time a function of accumulated rainfall? Explain your thinking.

Stronger and Clearer: Share your responses with 2–3 partners, to both give and receive feedback. Can you add more detail to your response? Use the feedback you receive to revise your response.

Log in to Amplify Math to complete this lesson online.

Name: .... Date: .... Period: ....

Activity 1 Moldy Bread

Clare took a break from her lab work for lunch and noticed mold on the last slice of bread in a plastic bag. As a scientist, she knew that mold was critical to the discovery of one of the greatest medical breakthroughs in history: penicillin, the first widely used antibiotic! Millions of lives have been saved, and continue to be saved, thanks in part to a little help from some mold.

The area covered by the mold on Clare's bread was about 1 mm^2. She left the bread alone to observe how the mold would grow. The next day, the area covered by the mold had doubled, and it doubled again the day after that.

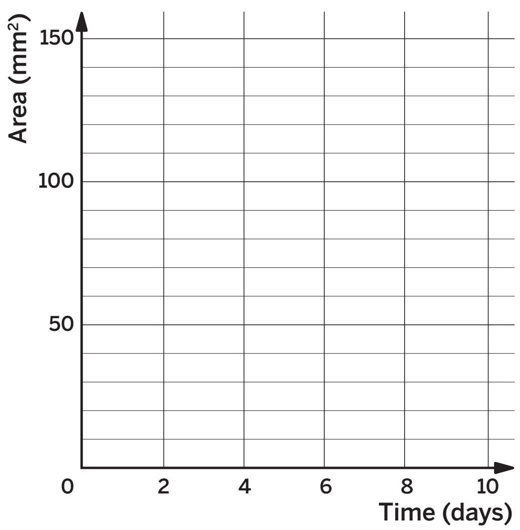
1. If the doubling pattern continues, how many square millimeters will the mold cover 4 days after she noticed the mold? Show your thinking.

2. Represent the relationship between the area A in square millimeters, covered by the mold and the number of days d since Clare first saw the mold using:

a A table of values, showing the area covered by the mold from the day Clare first saw it and over the next 5 days.

d	A

b A graph.



c An equation.

Activity 1 Moldy Bread (continued)

3. Discuss with your partner: Is the relationship between the number of days and the area covered by mold a function? If so, write "___ is a function of ___." If not, explain why it is not.

Are you ready for more?

What is an appropriate domain for the function representing the area of the mold? Explain your thinking.

Name: ________ Date: ________ Period: ________

Activity 2 Functionally Speaking

Here are some situations you have previously seen.

1. In a biology lab, a population of 50 bacteria reproduce by splitting. Every hour, on the hour, each bacterium splits into two bacteria.

   a. Write an equation that represents this scenario using function notation.

   b. Write a sentence of the form "____ is a function of ____."

   c. Indicate which is the independent variable and which is the dependent variable.

2. A new car is purchased for $\$18{,}000$. It loses $\frac{1}{3}$ of its value every year.

   a. Write an equation that represents this scenario using function notation.

   b. Write a sentence of the form "____ is a function of ____."

   c. Indicate which is the independent variable and which is the dependent variable.

3. To control an algae bloom in a lake, scientists introduce some treatment products. The day they begin treatment, the area covered by algae is 240 yd^2. Each day since the treatment began, $\frac{1}{3}$ of the previous day's area remains covered by algae. Time t is measured in days.

   a. Write an equation that represents this scenario using function notation.

   b. Write a sentence of the form "____ is a function of ____."

   c. Indicate which is the independent variable and which is the dependent variable.

Activity 3 Choose Limits for Your Axes

A team of scientists has been monitoring the amount of a new antibiotic in patients' bodies to see how long the medicine stays in their systems. The scientists have injected each patient with 20 mg of antibiotics. The equation $m = 20 \cdot (0.8)^h$ represents the amount of medicine m, in milligrams, left in a patient's body h hours after injection.

1. Complete the table to determine the amount of medicine in a patient's system.

h	m
	20
	12.8
	2.14
	0.28

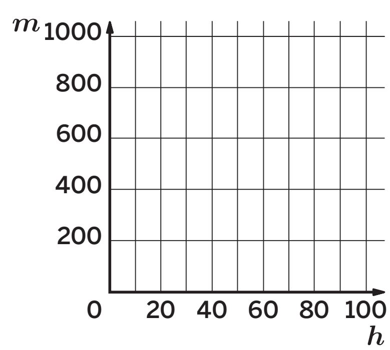
a Does this scenario represent a function? Explain your thinking.

b If the equation is a function, write it using function notation.

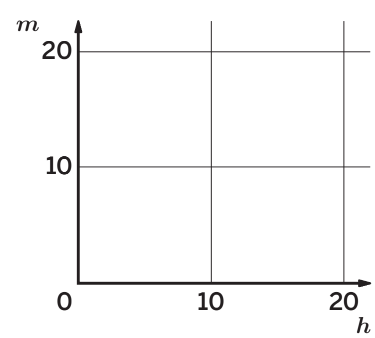
2. Without using a graphing tool, determine whether these inequalities represent the appropriate axes limits for this scenario. Explain your thinking.

$-10 < h < 100$ $\quad\quad$ $-100 < m < 1000$

3. Use graphing technology to verify your response. Set the graph with these axes limits: $-10 < h < 100$ and $-100 < m < 1000$. Sketch the graph. Do these limits help to identify specific information about the amount of antibiotics in the patient's system? Explain your thinking.



4. Change the axes limits so the graph is clearly displayed. Sketch the graph.



5. Compare the graphs in Problems 3 and 4. Explain why the axes limits for Problem 4 are more reasonable for this context.

STOP

Name: .... Date: .... Period: ....

Summary

In today's lesson . . .

You studied situations that are characterized by exponential change. These situations can be seen as functions. In each situation, there is a quantity — an *independent variable* — that determines another quantity — the *dependent variable*. They are functions because each value of the *independent variable* results in one and only one value of the *dependent variable*. Functions that describe exponential change are called ***exponential functions***.

An exponential function is of the form $f(x) = a \cdot b^x$.

Reflect:

Practice

Name: ______________________ Date: __________ Period: __________

1. A scientist measures the height h of a tree each month, and m is the number of months since the scientist first measured the height of the tree.

a Is the height h a function of the month m? Explain your thinking.

b Is the month m a function of the height h? Explain your thinking.

2. A bacteria population is $10{,}000$. It triples each day.

a Explain why the bacteria population b is a function of the number of days d because it was measured at $10{,}000$.

b Write an equation relating b and d.

3. For an experiment, a scientist designs a can, 20 cm in height, that can hold water. A tube is installed at the bottom of the can, allowing water to drain out. At the beginning of the experiment, the can is full. When the experiment starts, the water begins to drain, and $\frac{1}{3}$ of the water's height remains each minute.

a Explain why the height of the water in the can is a function of time.

b The height h in centimeters, is a function of time t, in minutes since poking the hole. Define the function $f(t) = h$ by writing its equation in function notation.

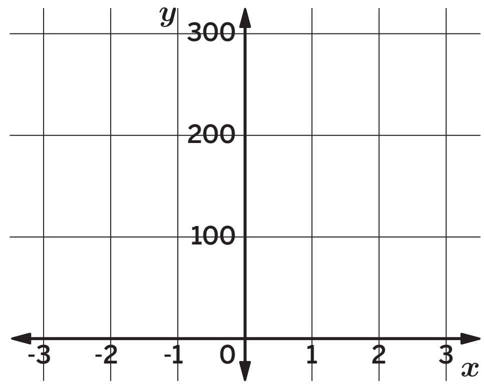
c Find and record the values for f when t is $0, 1, 2, 3$, and 4. What does $f(4)$ represent?

d What happens to the level of water in the can as time continues to elapse? How would this appear in a graph?

Practice

Name: .... Date: .... Period: ....

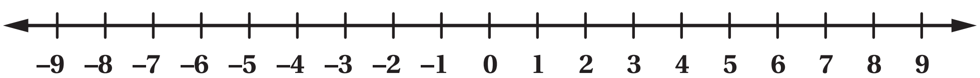
4. The graph shows an exponential function.



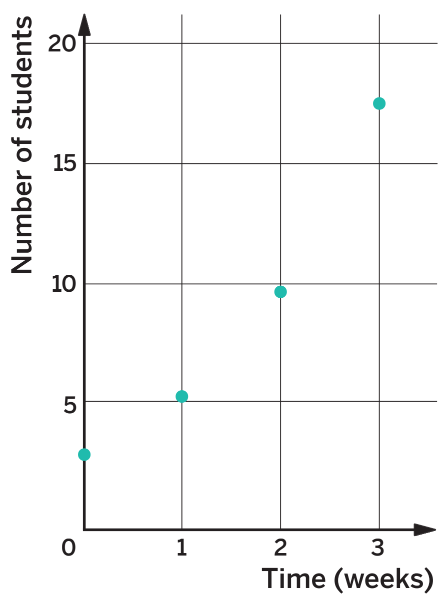
a Write an equation representing the relation.

b What is the value of y when $x = -1$?

5. Graph the solution set to the inequality $3x - 1 > 34 - 4x$ on the number line.



6. The number of students in an elementary school with chickenpox is modeled by the function $f(w) = 3 \bullet (1.8)^w$ of the number of weeks w since the school nurse first discovered the outbreak.



a According to the model, how many students did the school nurse initially discover had chickenpox? Explain your thinking.

b According to the model, how quickly is the chickenpox spreading?

c What does the $f(3)$ mean in this scenario?

Unit 4 | Lesson 11

Interpreting Exponential Functions

Let's find some meaningful ways to represent exponential functions.

Warm-up Equivalent or Not?

Lin and Diego are discussing two expressions: x^2 and 2^x.

- **Lin says, "I think the two expressions are equivalent."**
- **Diego says, "I think the two expressions are only equal for some values of x."**

Do you agree with either of them? Explain your thinking.
Complete the table to help you decide.

x	1	2	3	4	5
x^2	1				
2^x	2				

Log in to Amplify Math to complete this lesson online.

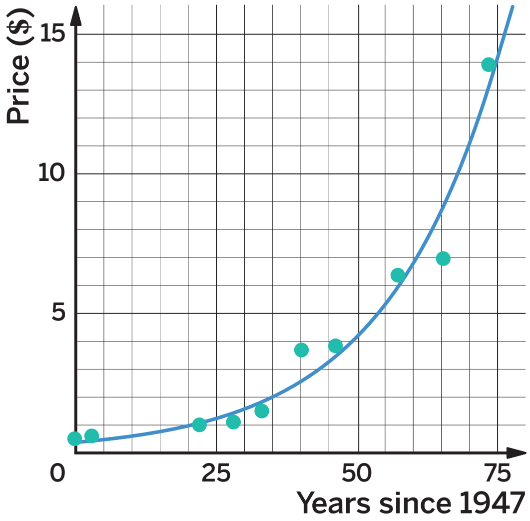
Name: .... Date: .... Period: ....

Activity 1 Cost of a Bottle of Aspirin

The ancient Greeks were known to have used the bark and leaves of the willow tree, which contains salicin, for medicinal purposes. Salicin was refined into salicylic acid in the early 1800s. Today, it is further refined into acetylsalicylic acid, or aspirin, one of the most widely used analgesics in history!

WalterCicchetti/Shutterstock.com

The price, in dollars, of a bottle of aspirin can be modeled by an exponential function $f(t)$ where t is the number of years since 1947.



1. What does the statement $f(6) \approx 0.5$ say about this situation?

2. What is $f(35)$? What about $f(72)$? What do these values represent in this context?

3. When $f(t) = 8$, what is t? What does $f(t) = 8$ represent in this context?

Activity 2 Paper Folding

You will receive a sheet of paper. An unfolded sheet of paper is 0.1 mm thick. Fold the paper in half, and continue folding it in half as many times as you can.

1. Estimate the paper's thickness in millimeters after each fold. Record your measurements in the table below.

Number of folds	0	1	2	3	4	5	6
Thickness (mm)	0.1						

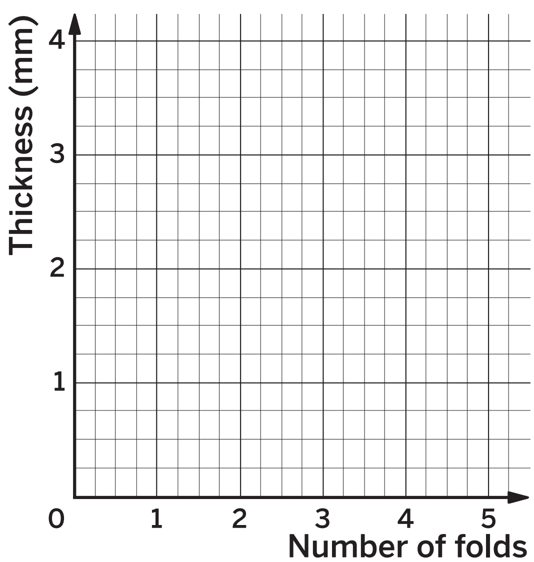
Did you expect the thickness to grow exponentially with each fold? Why or why not?

2. A student measures the thickness t, in millimeters, of a folded sheet of paper after it is folded n times. The student finds that t is given by the equation $t = (0.1) \cdot 2^n$.

a What does the number 0.1 represent in the equation?

b Using graphing technology, graph the equation $t = (0.1) \cdot 2^n$. Sketch the graph.

c How many folds will it take until the folded sheet of paper is more than 1 mm thick? How many folds until it is more than 1 cm thick? Explain your thinking.



Name: ______ Date: ______ Period: ______

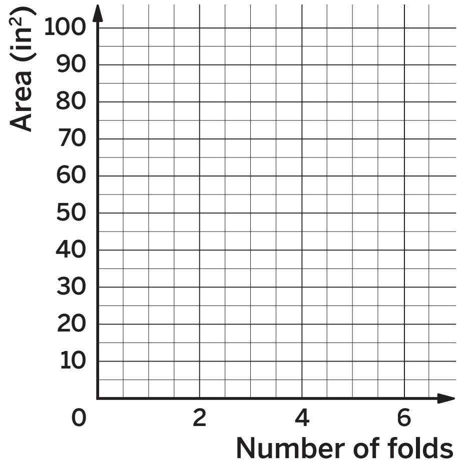
Activity 2 Paper Folding (continued)

3. The area of a sheet of paper is 93.5 in^2.

a Determine the area of the top face of the paper after it is folded in half once, in half twice, and in half three times.

b Write an equation for the area A of the top face of the paper in terms of the number of times it has been folded n.

c Use graphing technology to graph your equation. Sketch the graph.



d In this context, can n have negative values? Explain your thinking.

e Can A have negative values? Explain your thinking.

Are you ready for more?

1. How many folds are needed to reach 1 m in thickness? 1 km in thickness? Explain your thinking.

2. Do some research: What is the current world record for the number of times humans were able to fold a sheet of paper?

Activity 3 Info Gap: Smartphone Sales

You will receive either a data card or a problem card. Do not show or read your card to your partner. Match the data card with its corresponding problem card.

If you have the data card:	If you have the problem card:
1. Silently read the information on your card.	**1.** Silently read your card and think about what information you need to solve the problem.
2. Ask your partner, "What specific information do you need?" and wait for your partner to ask for information.	**2.** Ask your partner for the specific information that you need.
3. Before telling your partner the information, ask, "Why do you need to know (that piece of information)?"	**3.** Explain to your partner how you are using the information to solve the problem.
4. Read the problem card, and solve the problem independently.	**4.** When you have enough information, share the problem card with your partner, and solve the problem independently.
5. Share the data card, and discuss your reasoning.	**5.** Read the data card, and discuss your reasoning.

Pause here so your teacher can review your work. Ask your teacher for a new set of cards and repeat the activity, trading roles with your partner.

STOP

Name: ______ Date: ______ Period: ______

Summary

In today's lesson . . .

You analyzed graphs of exponential functions, such as $A = 93.5 \bullet \left(\frac{1}{2}\right)^n$, and described the relationships using function notation. In this example, the area A of the paper was a function of number of folds n. You also used exponential functions to solve problems about different real-world situations and determined when it made sense to connect the discrete points on a graph with a curve.

Reflect:

Practice

Name: .... Date: .... Period: ....

1. The number of people with the flu during an epidemic is a function of the number of days d since the epidemic began. The function $f(d) = 50 \cdot \left(\frac{3}{2}\right)^d$ defines this relationship.

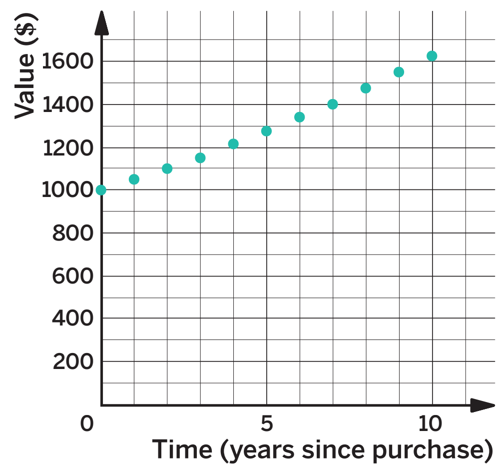
a. How many people had the flu at the beginning of the epidemic? Explain your thinking.

b. How quickly is the flu spreading? Explain how you can tell from the equation.

c. What does $f(1)$ mean in this situation?

d. Does $f(3.5)$ make sense in this situation?

2. The function $f(t)$ gives the dollar value of a bond t years after the bond was purchased. The graph of $f(t)$ is shown.



a. What is $f(0)$? What does it mean in this situation?

b. What is $f(4.5)$? What does it mean in this situation?

c. When is $f(t) = 1500$? What does this mean in this situation?

3. *Graphing technology required.* A model for the number of stray cats in a town t years since the town started an animal control program is the function $f(t) = 243 \cdot \left(\frac{1}{3}\right)^t$. The program includes both sterilizing stray cats and finding homes to adopt them.

a. What is the value of $f(t)$ when t is 0? Explain what this value means in this situation.

b What is the approximate value of $f(t)$ when t is $\frac{1}{2}$? Explain what this value means in this situation.

c What does the number $\frac{1}{3}$ in the model tell you about the stray cat population?

d Use graphing technology to graph $f(t)$ for values of t between 0 and 4. What axes limits allow you to see values of $f(t)$ that correspond to these values of t?

4. The function $g(t) = 600 \cdot \left(\frac{3}{5}\right)^t$ gives the amount of a chemical in a person's body, in milligrams, t hours since the patient took the medicine.

a What does the fraction $\frac{3}{5}$ mean in this situation?

b Sketch a graph of $g(t)$.

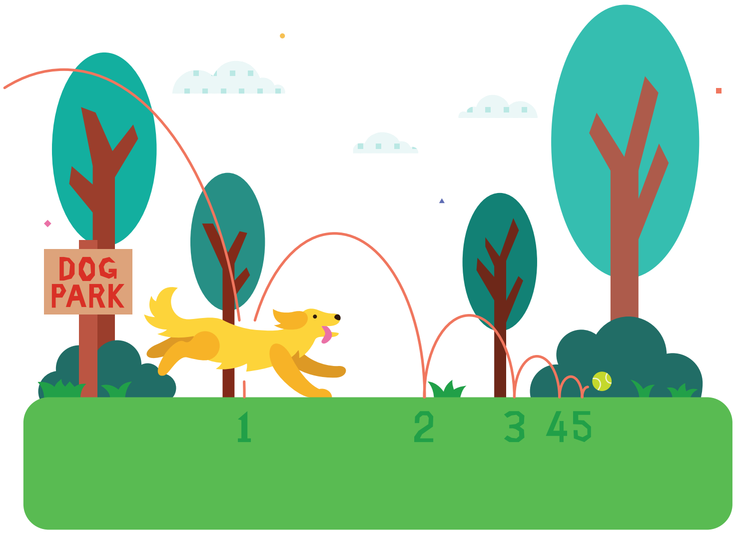
c What are the domain and range of $g(t)$? Explain what they mean in this situation.

5. Clare bought a moped vehicle for $\$2{,}500$. Every year since her purchase, the dollar value of the vehicle decreases by a half of it's dollar value. Is this scenario exponential or linear? Explain your thinking.

Unit 4 | Lesson 12

Modeling Exponential Behavior

Let's use exponential functions to model real-world situations.



Warm-up Linear or Exponential?

Circle whether each scenario is exponential or linear.

a A physician orders a patient to take 500 mg of a medicine every hour.

Exponential or Linear

b There are some bacteria in a dish. Every hour, each bacterium splits into 2 bacteria.

Exponential or Linear

c

Number of flu shots	Total cost ($)
1	5
2	10
4	20
5	25

Exponential or Linear

d

Time, t (hours)	Medicine remaining in the body, m (mg)
0	500
1	250
2	125
4	31.25

Exponential or Linear

e $f(x) = 2^x$

Exponential or Linear

f $f(x) = 2 \cdot x$

Exponential or Linear

Log in to Amplify Math to complete this lesson online.

Name: .... Date: .... Period: ....

Activity 1 Beholding Bounces

Here are the measurements for the maximum height of a tennis ball after bouncing several times on a concrete surface.

Bounce number, n	Height, h (cm)
0	150
1	80
2	43
3	20
4	11

1. Which is more appropriate for modeling the maximum height h, in centimeters, of the tennis ball after n bounces: a linear function or an exponential function? Use data from the table to support your response.

2. Regulations say that a tennis ball, dropped on concrete, should rebound to a height between 53% and 58% of the height from which it is dropped. Does this tennis ball meet the requirement? Explain your thinking.

3. Write an equation that models the bounce height h after n bounces of this tennis ball.

4. About how many bounces will it take before the rebound height of the tennis ball is less than 1 cm? Explain your thinking.

Activity 2 Beholding More Bounces

The table shows some heights of a ball after a certain number of bounces. Some heights are missing.

Bounce number	Height (cm)
0	?
1	?
2	73.5
3	51.5
4	36

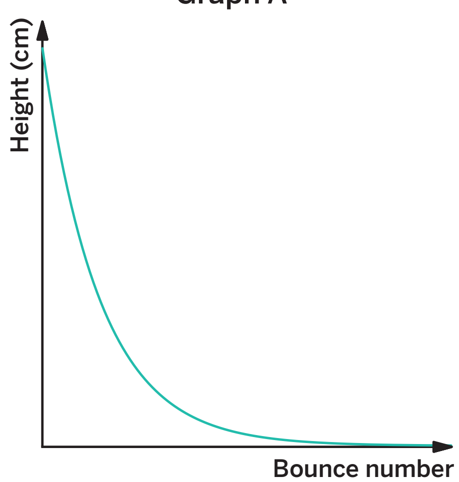
1. Is this ball more or less bouncy than the tennis ball in Activity 1? Explain your thinking.

2. From what height do you think the ball was dropped? Explain or show your thinking.

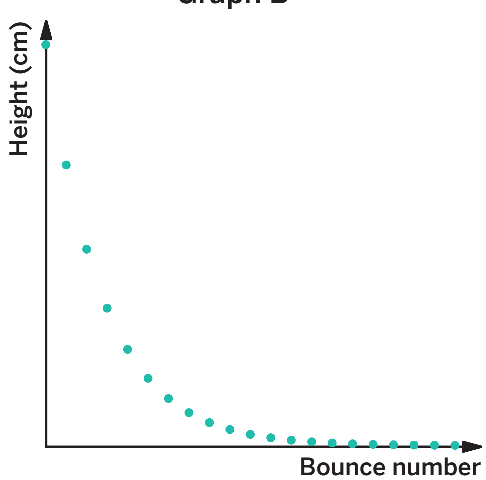
3. Write an equation that represents the bounce height of the ball h, in centimeters, after n bounces.

4. Which graph would more appropriately represent the equation for h: Graph A or Graph B? Explain your thinking.

Graph A



Graph B

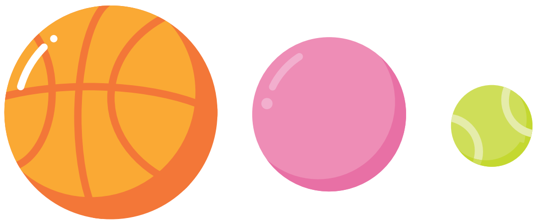


5. Will the nth bounce of this ball be lower than the nth bounce of the tennis ball? Explain your thinking.

Name: ______ Date: ______ Period: ______

Activity 3 Which Is the Bounciest of All?

Your group will receive three different kinds of balls. Your goal is to measure the rebound heights, model the relationship between the number of bounces and the heights, and compare the bounciness of the balls.



1. Complete the table. Make sure to note which ball goes with which column.

Number of bounces, n	Height for Ball 1, a (cm)	Height for Ball 2, b (cm)	Height for Ball 3, c (cm)
0	150	150	150
1			
2			
3			
4			

2. Which one appears to be the bounciest? The least bouncy? Explain your thinking.

3. For each ball, write an equation expressing the bounce height in terms of the bounce number.

Activity 3 Which Is the Bounciest of All? (continued)

4. Explain how the equations tell you which ball is the most bouncy.

5. If the bounciest ball was dropped from a height of 300 cm, what equation would model its bounce height?

Are you ready for more?

Use the data you collected to respond to the following problems.

1. If Ball 1 was dropped from a point that was twice as high, would its bounciness be *greater*, *less*, or *the same*? Explain your thinking.

2. Ball 4 is half as bouncy as the least bouncy ball. What equation would describe its height h in terms of the number of bounces n?

3. Ball 5 was dropped from a height of 150 cm. It bounced up very slightly once or twice and then began rolling. How would you describe its rebound factor? Explain your thinking.

STOP

Name: ______ Date: ______ Period: ______

Summary

In today's lesson . . .

You explored successive bounce heights of different kinds of balls, represented as tables.

From these tables, you wrote exponential functions and interpreted the decay factor and initial value in terms of the context. The independent variable (the number of bounces) was discrete, so your data was best represented by a discrete graph. You also determined if tennis balls fell within regulation based on their rebound factor.

Despite the messiness of real-world data like this, you can distinguish between linear and exponential data, and model the data appropriately.

Reflect:

Practice

Name: ________ Date: ________ Period: ________

1. Which equation is most appropriate for modeling the data in the table?

x	1	2	3	4	5	6
y	79	101	124	158	194	244

A. $y = 64 \cdot (1.25)^x$

B. $y = 79 \cdot (1.25)^x$

C. $y = 79 + 1.25x$

D. $y = 64 + 22x$

2. The function $h(n) = 120 \cdot \left(\frac{4}{5}\right)^n$ describes the height of a ball, in inches, after n bounces.

a. What is $h(3)$? What does it represent in this scenario?

b. Could $h(n)$ be 150? Explain your thinking.

c. Which ball loses its bounce height more quickly, this ball or a tennis ball whose height in inches after n bounces is modeled by the function $f(n) = 50 \cdot \left(\frac{5}{9}\right)^n$?

d. How many bounces would it take before this ball bounces less than 12 in. from the surface?

3. The table shows the number of employees and number of active customer accounts for a few different marketing companies. Would a linear or exponential model for the relationship between number of employees and number of customers be more appropriate? Explain your thinking.

Number of employees	Number of customers
1	4
2	8
3	13
4	17
10	39

Name: ________ Date: ________ Period: ________

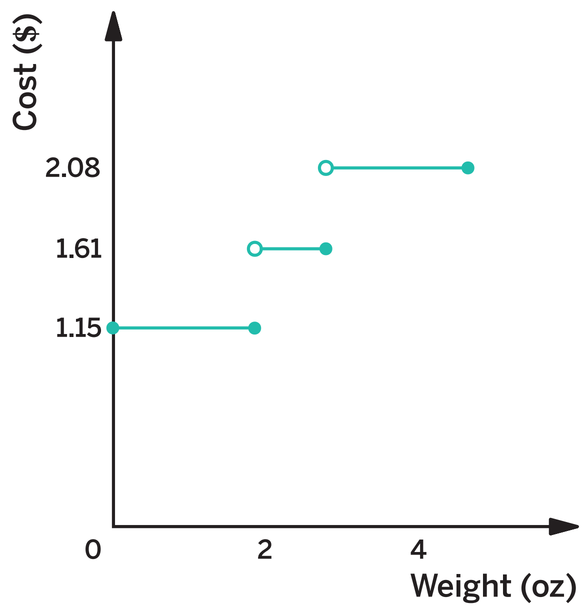
Practice

4. A bank account has a balance of $\$1{,}000$ dollars. It grows by a factor of 1.04 each year.

   a. Explain why the balance, in dollars, can be described by a function f of the number of years t since the account was opened.

   b. Write an equation defining f.

5. The graph shows the cost in dollars of mailing a letter from the United States to Canada in 2018 as a function of its weight in ounces.



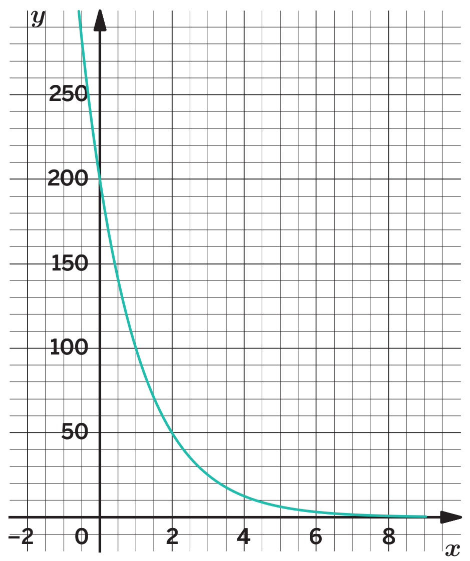
   a. How much does it cost to send a letter that weighs 1.5 oz?

   b. How much does it cost to send a letter that weighs 2 oz?

   c. What is the range of this function?

6. The equation for the function $f(x)$ and the graph of $y = g(x)$ are shown. What do you notice about the relationship between the equation and graph of the function?

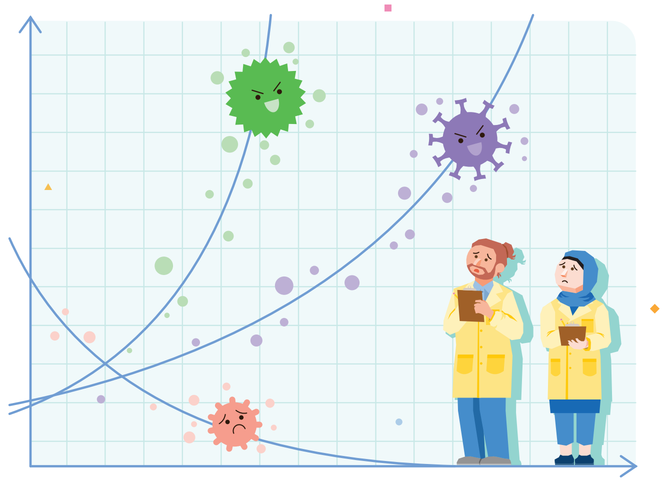
$$f(x) = 200\left(\frac{1}{2}\right)^x$$



Unit 4 | Lesson 13

Reasoning About Exponential Graphs

Let's study and compare equations and graphs of exponential functions.



Warm-up Spending Gift Money

Jada received a gift of $180. In the first week, she spent a third of the gift money. Each week, she continued to spend a third of what was left.

Which equation best represents the amount of gift money g, in dollars, she has after t weeks? Explain your thinking.

A. $g = 180 - \frac{1}{3}t$

B. $g = 180 \cdot \left(\frac{1}{3}\right)^t$

C. $g = \frac{1}{3} \cdot 180^t$

D. $g = 180 \cdot \left(\frac{2}{3}\right)^t$

Log in to Amplify Math to complete this lesson online.

Name: ______ Date: ______ Period: ______

Activity 1 Equations and Their Graphs

The following problems involve exponential functions of the form $f(x) = a \cdot b^x$.

1. The four functions f, g, h, and j represent the number of people infected by four different viruses over the course of x days.

   $f(x) = 50 \cdot 2^x$ $\quad g(x) = 50 \cdot 3^x$ $\quad h(x) = 50 \cdot \left(\frac{3}{2}\right)^x$ $\quad j(x) = 50 \cdot (0.5)^x$

   a. Use graphing technology to plot each function on the same coordinate plane, and then sketch the graph.

   b. Explain how changing the value of b changes the spread of each virus. Which virus is spreading the fastest?

2. The functions p, q, and r represent the number of people in three neighboring cities affected by a recent outbreak of contaminated drinking water, over the course of x days.

   $p(x) = 10 \cdot 4^x$ $\quad q(x) = 40 \cdot 4^x$ $\quad r(x) = 100 \cdot 4^x$

   a. Use graphing technology to graph each function on the same coordinate plane, and then sketch the graph.

   b. Explain how changing the value of a changes the severity of the outbreak and number of those affected in each city.

   c. Which city will have the most affected people after one year? How can you determine this without doing any calculations? Explain your thinking.

Are you ready for more?

Consider the following functions, which represent the amount of money in your bank account, in dollars, as a function of the time x in years: $\quad f(x) = 10 \cdot 3^x \quad h(x) = \frac{1}{2} \cdot 3^{x+3}$

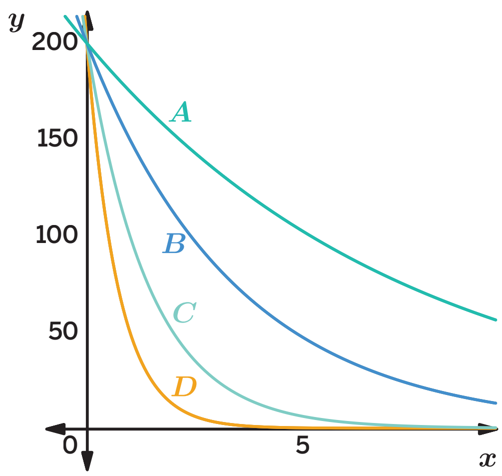
If you could choose one of these functions, which would it be? Does your choice depend on x? Explain your thinking.

Activity 2 Graphs of Exponential Decay

The following four functions represent the amount of an antibiotic medicine, in milliliters, left in the body as a function of the number of days, x. For each antibiotic, 200 ml are initially administered.

$m(x) = 200 \cdot \left(\frac{1}{4}\right)^x$ $n(x) = 200 \cdot \left(\frac{1}{2}\right)^x$ $p(x) = 200 \cdot \left(\frac{3}{4}\right)^x$ $q(x) = 200 \cdot \left(\frac{7}{8}\right)^x$

1. Match each equation with a graph. Explain your thinking.



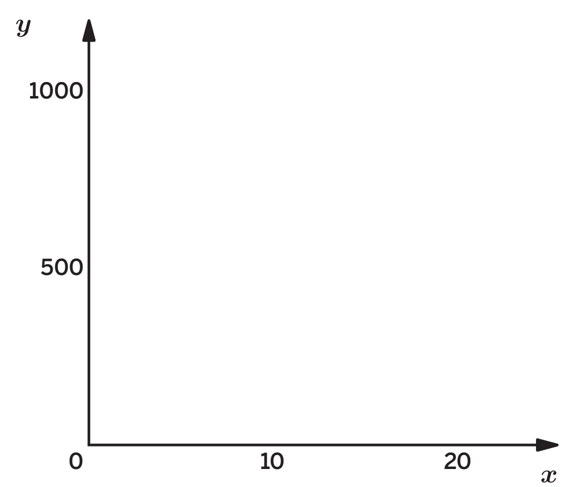
2. Two new antibiotics just entered the market. The following functions represent the amount, in milliliters, left in the body as a function of the number of days x.

$f(x) = 1000 \cdot \left(\frac{1}{10}\right)^x$ $g(x) = 1000 \cdot \left(\frac{9}{10}\right)^x$

a. What is the initial amount of these antibiotics administered to patients?

b. Which of these antibiotics exits the patients' bodies most quickly? Explain your thinking.

c. Use graphing technology to verify your response, and sketch your graph.



STOP

Name: ______________________ Date: __________ Period: __________

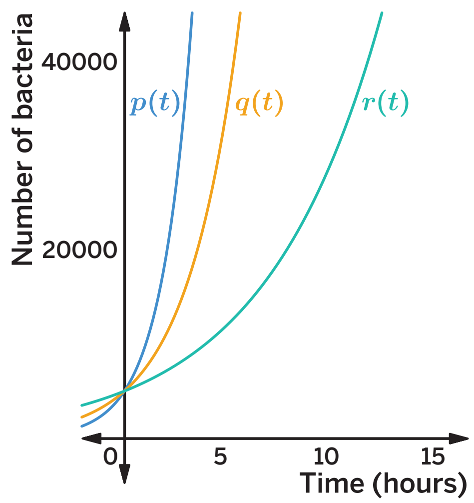
Summary

In today's lesson . . .

You saw that an exponential function can give you information about a graph that represents it.

For example, suppose the function $q(t) = 5000 \cdot (1.5)^t$ represents a bacteria population, t hours after it is first measured.

A graph can help you see how the starting population, $5{,}000$, and growth factor, 1.5, influence the population. Suppose the functions $p(t) = 5000 \cdot 2^t$ and $r(t) = 5000 \cdot (1.2)^t$ represent two other bacteria populations. Here are the graphs of p, q, and r.



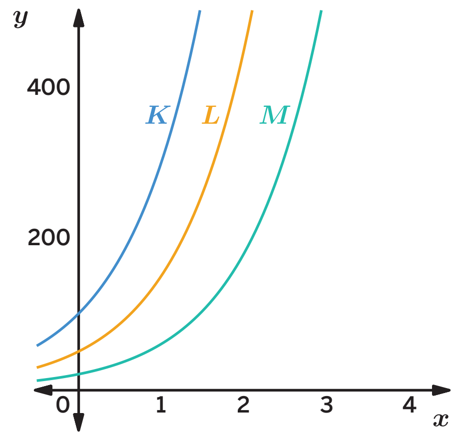
All three graphs start at $5{,}000$, but the graph of r grows slower than the graph of q, while the graph of p grows the fastest. This makes sense, because a population that doubles every hour grows more quickly than one that increases by a factor of 1.5 each hour. Both grow more quickly than a population that increases by a factor of 1.2 each hour.

Reflect:

Practice

Name: .... Date: .... Period: ....

1. Here are graphs of three exponential equations. Match each equation with its graph.



a. $y = 20 \bullet 3^x$

b. $y = 50 \bullet 3^x$

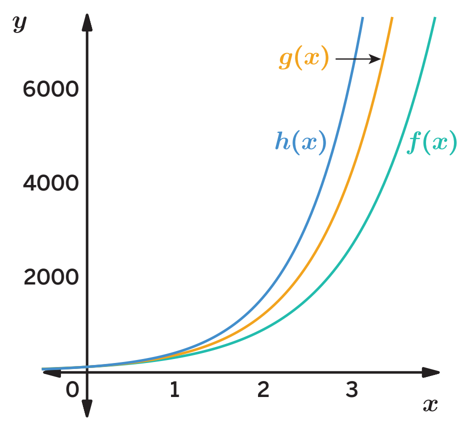
c. $y = 100 \bullet 3^x$

2. Here are three exponential functions f, g, and h.

$f(x) = 100 \bullet 3^x$

$g(x) = 100 \bullet (3.5)^x$

$h(x) = 100 \bullet 4^x$

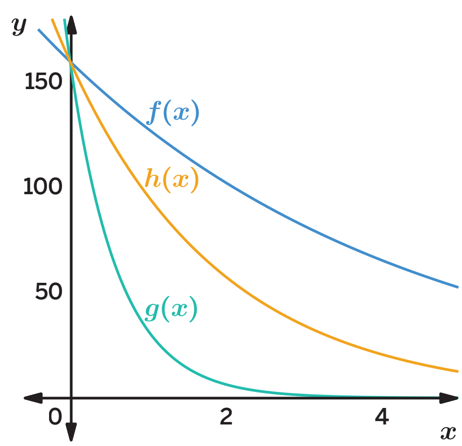


a. Which of these functions grows the slowest? The fastest? Explain your thinking.

b. Why do all three graphs share the same intersection point with the vertical axis?

3. The functions $f(x) = 160 \bullet \left(\frac{4}{5}\right)^x$ and $g(x) = 160 \bullet \left(\frac{1}{5}\right)^x$ are shown on the graph.

If the function h is defined by $h(x) = a \bullet b^x$, what can you say about the values of a and b? Explain your thinking.



Name: .... Date: .... Period: ....

Practice

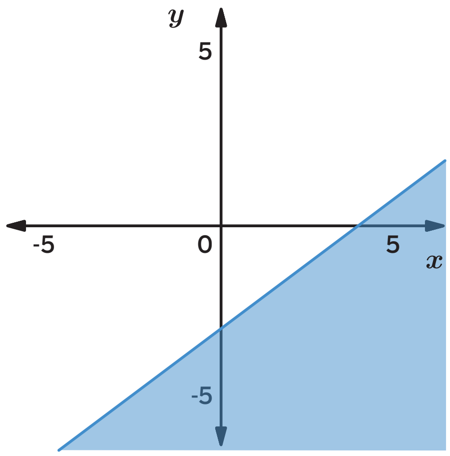
4. Select the inequality whose solution is represented by the graph.

A. $3x - 4y > 12$

B. $3x - 4y \geq 12$

C. $3x - 4y < 12$

D. $3x - 4y \leq 12$



5. Start with a square whose area is 1 square unit, Figure 1. Subdivide it into 9 squares of equal area, and remove the middle one to obtain Figure 2.

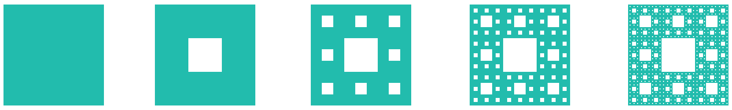


Figure 1 Figure 2 Figure 3 Figure 4 Figure 5

a What is the area of Figure 2?

b Take the remaining 8 squares, subdivide each of them into 9 equal squares, and remove the middle one from each. What is the area of Figure 3?

c Write an equation representing the area A of Figure n.

6. A hospital is flooded with 300 patients as a severe flu epidemic hits the city. The number of sick patients p is a function of the number of days d since the patients have received treatment. The table shows the values of p. Calculate the average rate of change for the following intervals:

d	0	2	4	6
$p(d)$	300	213	142	103

a Day 0 to Day 2.

b Day 2 to Day 6.

Unit 4 | Lesson 14

Looking at Rates of Change

Let's calculate average rates of change for exponential functions.

Warm-up Falling Prices

The function $p(t)$ gives the cost, in dollars, of producing a solar panel capable of generating 1 watt of power, t years after 1977. Here is a table showing the values of p from 1977 to 1987.

Which expression best represents the average rate of change in solar cost between 1977 and 1987?

A. $p(10) - p(0)$

B. $p(10)$

C. $\dfrac{p(10) - p(0)}{10 - 0}$

D. $\dfrac{p(10)}{p(0)}$

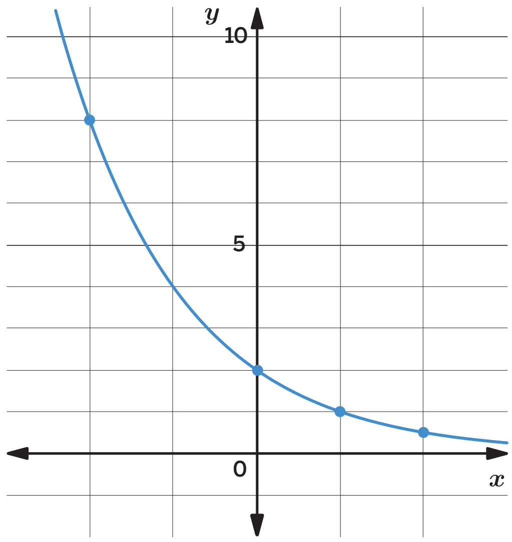
t	$p(t)$
0	80
1	60
2	45
3	33.75
4	25.31
5	18.98
6	14.24
7	10.68
8	8.01
9	6.01
10	4.51

Log in to Amplify Math to complete this lesson online.

Name: ______ Date: ______ Period: ______

Activity 1 Average Rate of Change, Revisited

Consider the following graph of the function $f(x)$.



1. What is the equation of the graph of $f(x)$?

2. What is the average rate of change of $f(x)$ for the interval from $x = -2$ to $x = 0$?

3. What is the average rate of change of $f(x)$ for the interval from $x = -2$ to $x = 1$?

4. What is the average rate of change of $f(x)$ for the interval from $x = -2$ to $x = 2$?

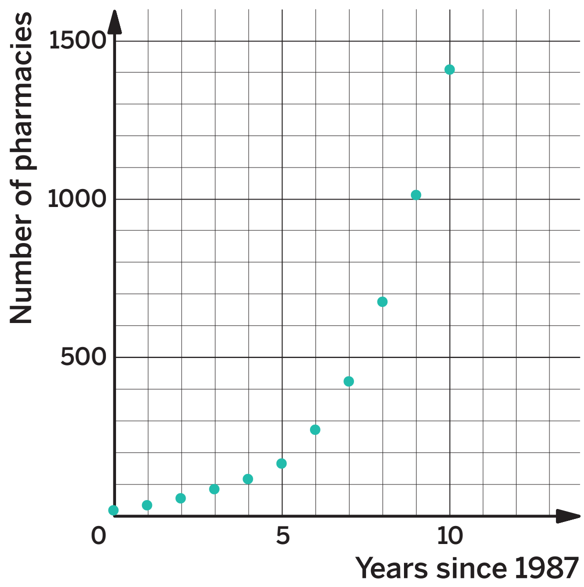
5. Draw a line through the points at the beginning and end of each interval in Problems 2, 3, and 4. What is the slope of each line?

6. Which of these average rates of change best represents the change in $f(x)$ at point $(-1, 4)$? Explain your thinking.

Activity 2 Further Pharmacy Expansion

The table and graph show the number of retail pharmacies worldwide that a company had in its first 10 years, between 1987 and 1997. The growth in the number of retail pharmacies was approximately exponential.

Year	Number of pharmacies
1987	17
1988	33
1989	55
1990	84
1991	116
1992	165
1993	272
1994	425
1995	677
1996	1,015
1997	1,412



Co-craft Questions: Before you begin Problem 1, study the table and graph. Work with your partner to write 2–3 mathematical questions you have about this scenario.

1. Find the average rate of change for each period of time. Show your thinking.

a. From 1987 to 1990

b. From 1987 to 1993

c. From 1987 to 1997

Name: ____________ Date: ____________ Period: ____________

Activity 2 Further Pharmacy Expansion (continued)

2. What do you observe about the average rates of change you calculated? What do they tell you about how the company was growing during this time?

3. On the graph, draw a line to represent the average rate of change in the first 3 years. Does this line fit the data? How well does this line describe the company's growth?

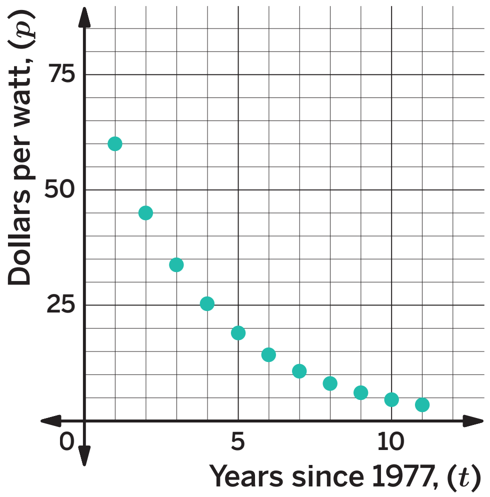
4. On the graph, draw a line to represent the average rate of change in the first 6 years. Does this line fit the data? How well does this line describe the company's growth?

5. On the graph, draw a line to represent the average rate of change over the entire 10 years. Does this line fit the data? How well does the line describe the growth of the company?

6. The function $f(t)$ represents the number of retail pharmacies t years since 1987. The value of $f(20)$ is $15{,}011$. Determine $\frac{f(20) - f(10)}{20 - 10}$ and describe what it tells you about the change in the number of pharmacies.

Activity 3 Cost of Solar Cells

The graph shows the exponential function $p(t)$, which models the cost, in dollars, of producing a solar panel capable of generating 1 watt of power, from 1977 to 1988 where t is the number of years since 1977.



1. Bard said, "Over the first five years, between 1977 and 1982, the cost fell by about $\$12$ per year. But in the second five years, between 1982 and 1987, the cost fell only by about $\$3$ a year." Show that Bard is correct.

2. If the trend continues, will the average decrease in price be more or less than $\$3$ per year between 1987 and 1992? Explain your thinking.

Are you ready for more?

Suppose the cost of producing a solar panel that generated 1 watt had instead decreased by $\$12.20$ each year between 1977 and 1982. Compute what the costs would be each year, and plot them on the same graph shown in the activity. How do these alternate costs compare to the actual costs shown?

STOP

Name: ____________ Date: ________ Period: ________

Summary

In today's lesson . . .

You explored the average rates of change for exponential functions over specified intervals. For linear functions, the average rate of change is the same no matter which interval is chosen. A constant rate of change is a key feature of linear functions; when represented graphically, the slope of the line *is* the rate of change.

But what about the average rate of change for an exponential function?

The table shows how many square yards $A(t)$ of algae remain t weeks since treatment began on a pond to control its algae bloom.

t	$A(t)$
0	240
1	80
2	27
3	9
4	3

The average rate of change from Week 0 to Week 2 is about -107 yd^2 per week: $\frac{A(2) - A(0)}{2 - 0} \approx -107$.

The average rate of change from Week 2 to Week 4 is only about -12 yd^2 per week: $\frac{A(4) - A(2)}{4 - 2} = -12$.

These calculations show that $A(t)$ is decreasing over both intervals, but the decrease is greater from Weeks 0–2 than from Weeks 2–4.

Reflect:

Practice

Name: ______________________ Date: __________ Period: __________

1. A store receives $2{,}000$ decks of popular trading cards. The number of decks d is a function of the number of days t since the shipment arrived. Here is a table showing some values of d.

t	0	5	10	15	20
$d(t)$	2,000	1,283	823	528	338

Calculate the average rate of change over the following intervals.

a. Day 0 to Day 5

b. Day 15 to Day 20

2. A study was conducted to analyze the effects on deer population in a particular area. The function $f(t) = a \bullet b^t$ gives the population of deer t years after the study began. If the population is increasing, select *all* statements that must be true.

A. $b > 1$

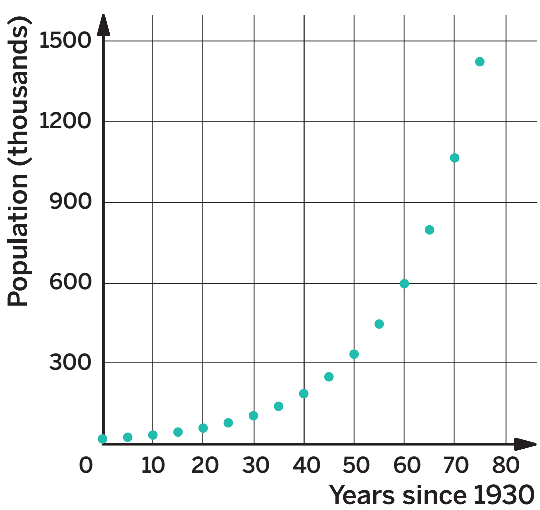
B. $b < 1$

C. The average rate of change from Year 0 to Year 5 is less than the average rate of change from Year 10 to Year 15.

D. The average rate of change from Year 0 to Year 5 is greater than the average rate of change from Year 10 to Year 15.

E. $a > 0$

3. The function f models the population, in thousands, of a city t years after 1930. The average rate of change from $t = 0$ to $t = 70$ is approximately $14{,}000$ people per year. Is this value a good way to describe the population change of the city over that time period? Explain your thinking.



Name: .... Date: .... Period: ....

4. The function $f(w) = 500 \cdot 2^w$ gives the number of copies a book has sold w weeks after it was published. Select *all* the domains for which the average rate of change would best represent the number of books sold.

A. $0 \leq w \leq 2$

B. $0 \leq w \leq 7$

C. $5 \leq w \leq 7$

D. $5 \leq w \leq 10$

E. $0 \leq w \leq 10$

5. *Graphing technology required.* A moth population p is modeled by the equation $p = 500000 \cdot \left(\frac{1}{2}\right)^w$, where w is the number of weeks since the population was first measured.

a. What was the moth population when it was first measured?

b. What was the moth population after 1 week? After 1.5 weeks?

c. Use technology to graph the population and determine when it falls below $10{,}000$.

6. Mai has two options to receive a pay raise. She currently makes 15 dollars per hour and can either receive a 10% increase on her hourly rate or a raise of 1 dollar per hour. Which should she choose? Explain your thinking.

My Notes:

SUB-UNIT

4 Percent Growth and Decay

Want to be CEO for a day?

Imagine being the CEO of a biotech company. Sure, there may be fancy suits, luxurious boardrooms, and a small army of assistants at your beck and call.

But it's also a high-stakes job overseeing the development of new medications, vaccines, and medical devices. To date, the global pharmaceutical industry is worth almost $1 trillion. And when people's lives depend on your company's products, difficult questions can come up.

Which diseases should you target?

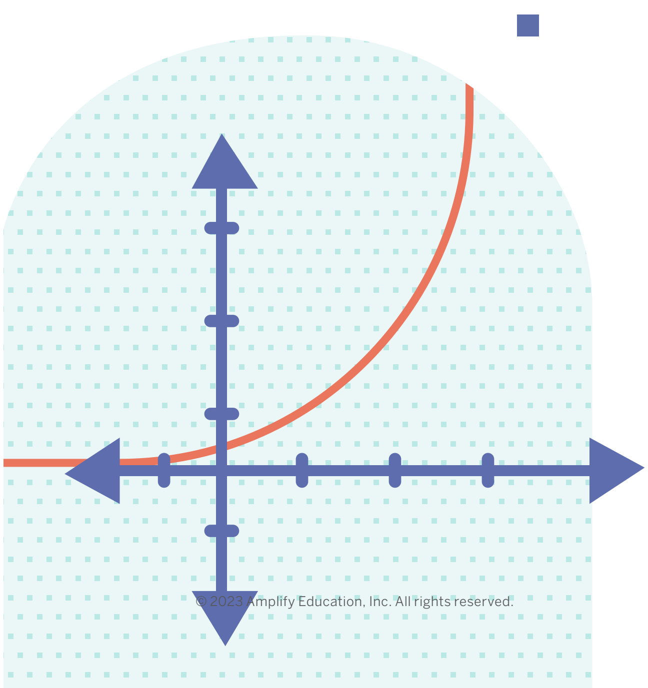
How much should a life-saving drug cost?

How do you help your company grow?

What is your responsibility to your shareholders?

These questions can be a heavy burden. Without understanding what the answers might mean — in terms of dollars and cents, market share captured, or lives saved — how would you know you made the best choice?

Whether it is the effect of a new drug on a patient population or the impact on your company's share price, being able to calculate *rates of change* gives you the power to plan ahead and make smart decisions for both your boardroom and your community.



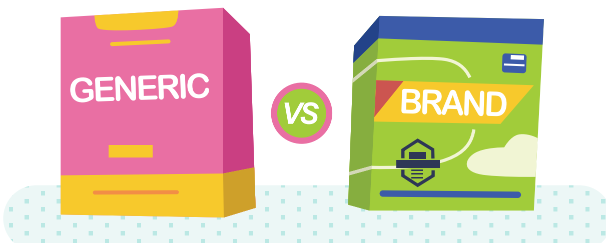
Unit 4 | Lesson 15

Recalling Percent Change

Let's see what happens when you change a number by a percentage.

Warm-up Same Symptoms, Different Prices

For each problem, show your thinking.



1. The cost of a generic brand of medication is 60% of the cost of a similar commercial brand. If the commercial brand costs \$15.99, what is the price of the generic brand?

2. The price of a commercial brand of medication is 30% more than a similar generic brand. If the generic brand costs \$14, what is the price of the commercial brand?

3. A generic prescription of medication costs 35% less than a similar commercial brand. The commercial brand costs \$49. How much does the generic brand cost?

Log in to Amplify Math to complete this lesson online.

Name: .... Date: .... Period: ....

Activity 1 Taxes and Sales

Complete the following problems. Describe how you would compute the costs, given the percentages in each scenario.

1. You need to pay 8% sales tax on a car that costs $\$12{,}000$. What will you end up paying in total? Show or explain your thinking.

2. Burritos are on sale for 30% off. Your favorite burrito normally costs $\$8.50$. How much does it cost now? Show or explain your thinking.

3. A pair of shoes that originally cost $\$79$ are on sale for 35% off. Does the expression $0.65(79)$ represent the sale price of the shoes, in dollars? Show or explain your thinking.

4. A store-brand allergy medication costs 55% less than a similar commercial brand name. If the brand name costs $\$19.97$, how much does the store brand cost? Show or explain your thinking.

Are you ready for more?

What are some strategies for mentally adding 15% to the total cost of an item?

Activity 2 The Global Burden of Disease

According to an Our World in Data report, between 1990 and 2017 the global number of reported cases of communicable (i.e., contagious) diseases decreased significantly. During this period, non-communicable diseases, such as chronic illnesses, increased. The number of reported cases of both non-communicable and communicable diseases recorded in 1990 and 2017 are shown in the table.

Year	Non-communicable diseases	Communicable diseases
1990	1,107,288,573	1,185,165,322
2017	1,550,896,145	695,990,294

1. What was the total number of reported cases of non-communicable and communicable diseases combined in 1990? In 2017?

2. Did the total number of reported cases increase or decrease between 1990 and 2017? By how much?

3. Approximately what percent of all diseases were non-communicable in 1990? In 2017?

4. Did the percentage increase or decrease, and by how much?

5. Approximately what percent of all diseases were communicable in 1990? In 2017?

6. Did the percentage increase or decrease, and by how much?

Name: ______ Date: ______ Period: ______

Activity 3 Expressing Percent Increase and Decrease

Complete the table so that each row has a scenario and two different expressions that solve the problem from the scenario. The second expression should use only multiplication. Be prepared to explain how the two expressions are equivalent.

Description and question	Expression 1	Expression 2 (using only multiplication)
A one-night stay at a hotel in Anaheim, CA, costs \$160. Hotel room occupancy tax is 15%. What is the total cost of a one-night stay?	$160 + (0.15) \cdot 160$	
Teachers receive a 30% discount at a museum. An adult ticket costs \$24. How much would a teacher pay for admission into the museum?		$(0.7) \cdot 24$
Ten years ago, the population of a city was 842,000. The city now has 2% more people than it had then. What is the population of the city now?		
After a major hurricane, 46% of the 90,500 households on an island lost their access to electricity. How many households still have electricity?		
	$754 - (0.21) \cdot 754$	
Two years ago, the number of students in a school was 150. Last year, the student population increased 8%. This year, it increased about 8% again. What is the number of students this year?		

STOP

Summary

In today's lesson . . .

You reviewed different ways of expressing percent increase and decrease. While there were several equivalent expressions for each scenario, using the Distributive Property to express the percent change using only multiplication is an efficient method.

Here are two examples:

1. A town had a population of $200,000$ last year. Its population increased by 5% this year.

   $$200000 + (0.05) \cdot 200000 = 200000 \cdot (1 + 0.05) = 200000 \cdot (1.05)$$

2. A shopper receives a 20% discount on $\$140$ worth of groceries.

   $$140 - (0.20) \cdot 140 = 140 \cdot (1 - 0.20) = 140 \cdot (0.80)$$

In both scenarios, you can use a combination of addition or subtraction and multiplication to arrive at the same percent change as multiplying directly.

Reflect:

Name: ______ Date: ______ Period: ______

Practice

1. For each scenario, write an expression that can be used to solve the problem. The expression should only use multiplication.

   a. Lin's salary is $\$2{,}500$ per month. She receives a 10% raise. What is Lin's new salary, in dollars per month?

   b. A test had 40 questions. A student answered 85% of the questions correctly. How many questions did the student answer correctly?

   c. A telephone cost $\$250$. The sales tax is 7.5%. What was the cost of the telephone, including sales tax?

2. In June, a family used $3{,}500$ gallons of water. In July, they used 15% more water. Select *all* the expressions that represent how many gallons of water the family used in July.

   A. $3500 + 0.15 \cdot 3500$

   B. $3500 + 0.15$

   C. $3500 \cdot (1 - 0.15)$

   D. $3500 \cdot (1.15)$

   E. $3500 \cdot (1 + 0.15)$

3. Han's summer job paid him $\$4{,}500$ last summer. This summer, he will receive a 25% pay increase from the company. Write two different expressions that could be used to find his new salary, in dollars.

Name: .... Date: .... Period: ....

4. Military veterans receive a 25% discount on movie tickets that normally cost $\$16$. Explain why $16 \cdot (0.75)$ represents the price of a ticket using the discount.

5. A new car costs $\$15{,}000$ and the sales tax is 8%. Explain why $15000 \cdot (1.08)$ represents the cost of the car, including tax.

6. Use the number 225 as a starting point.

a Determine a 25% increase of 225.

b Determine another 25% increase of the new value from part a.

Name: ______ Date: ______ Period: ______

Unit 4 | Lesson 16

Functions Involving Percent Change

Let's investigate what happens when we repeatedly apply a percent increase to a quantity.

Warm-up ℞ Discounts

A local pharmacy is offering a promotion: 25% off all prescriptions. Priya's prescription normally costs $32. The cashier applies the promotion, and then takes an *additional* 25% off for a coupon that Priya has. If there is no sales tax, how much does Priya pay for her prescription? Show your work.

Log in to Amplify Math to complete this lesson online.

Activity 1 Cost of Prescriptions

During a severe allergic reaction, muscles can swell and breathing can become difficult. When someone experiences a reaction like this, a device that injects epinephrine can save their life. Epinephrine is a medication that relaxes the muscles, causing the lungs to open.

In 2007, the price of a single injection device was $\$50$. Since 2007, the price of this device has been increasing at a rate of 21% annually. This price increase threatens to make this life-saving medication unaffordable for many people.

1. How much did the device cost after one year, in 2008? Show your thinking.

2. Assuming the price continued to rise at the same rate, how much would the device cost after two years? After three years?

3. Write an expression for the price of the device after x years.

4. How much would the device cost in 2030?

Are you ready for more?

To determine the price of the device after three years (in 2010), a student started by writing:

[Year 2 Amount] + [Year 2 Amount](0.21) = [Year 3 Amount]
and ended with
$50(1.21)(1.21)(1.21)$ = [Year 3 Amount]

Does her final expression correctly reflect the price after three years? Explain or show your thinking.

Name: ________________ Date: ________ Period: ________

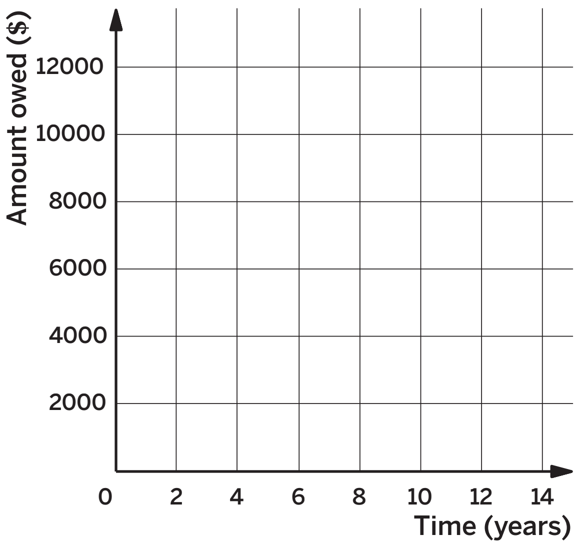
Activity 2 Comparing Loans

Suppose three people have each taken out loans of $1,000, but they each pay different annual interest rates.

1. For each loan, write an expression using only multiplication for the amount owed at the end of each year, if no payments are made.

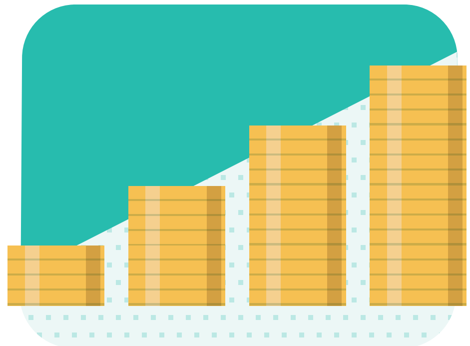
Years without payment	Loan A 12%	Loan B 24%	Loan C 30.6%
1			
2			
3			
10			
x			

2. Graph each loan. Based on your graph, approximately how many years would it take for the original unpaid balances of each loan to double?



Activity 3 Comparing Average Rates of Change

The functions a, b, and c represent the amount owed (in dollars) for Loans A, B, and C from Activity 2, respectively. The input for these functions is t, the number of years without payments.



1. For each loan, determine the average rate of change per year between:

   a. The start of the loan and the end of the second year.

   b. The end of the 10th year and the end of the 12th year.

2. How do the average rates of change for the three loans compare in each of the two-year intervals?

STOP

Name: ______________ Date: ______ Period: ______

Summary

In today's lesson . . .

You explored how to express repeated interest calculations. When you borrow money from a lender, the lender usually charges interest, a percentage of the borrowed amount as payment for allowing you to use the money. The interest is usually calculated at a regular interval of time (monthly, yearly, etc.).

Because exponential functions eventually grow very quickly, leaving a debt unpaid can be very costly. You saw that lower interest rates are better, and that when comparing interest rates, higher rates cause the debt to grow more quickly.

Reflect:

Practice

Name: .... Date: .... Period: ....

1. In 2011, the population of deer in a forest was 650.

a In 2012, the population increased by 15%. Write an expression, using only multiplication, that represents the deer population in 2012.

b In 2013, the population increased again by 15%. Write an expression that represents the deer population in 2013.

c If the deer population continues to increase by 15% each year, write an expression that represents the deer population t years after 2012.

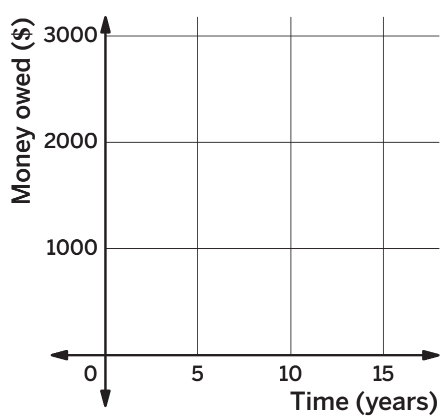
2. *Graphing technology required.* A $\$1{,}000$ loan charges 5% interest at the end of each year, while a second loan charges 8% interest at the end of each year. Assume that no payments are made and that the interest applies to the entire loan balance, including any previous interest charges.

a Complete the table with the balances for each loan.

Years, t	Loan balance, b (5% interest)	Loan balance, c (8% interest)
1		
2		
3		
t		

b Which loan balance grows more quickly? How will this be visible in the graphs of $b(t)$ and $c(t)$?

c Create graphs representing b and c over time. The graphs should show the starting balance of each loan, as well as the amount of the loan after 15 years.



Name: .... Date: .... Period: ....

Practice

3. The real estate tax rate in 2018 in a small rural county is increasing by $\frac{1}{4}$%. Last year, a family paid $1,200. Which expression represents the real estate tax, in dollars, that the family will pay this year?

A. $1200 + 1200 \cdot \frac{1}{4}$

B. $1200 \cdot (1.25)$

C. $1200 \cdot (1.025)$

D. $1200 \cdot (1.0025)$

4. Select *all* situations that are accurately described by the expression $15 \cdot 3^5$.

A. A population of bacteria begins at 15,000. The population triples each hour. How many bacteria are there after 5 hours?

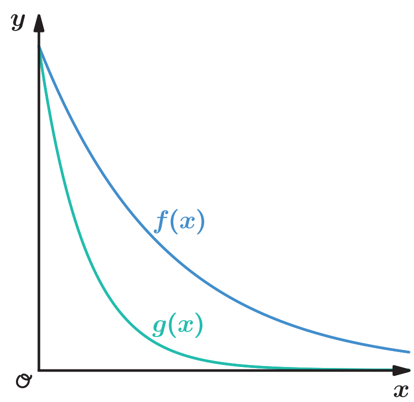
B. A population of bacteria begins at 15,000. The population triples each hour. How many thousands of bacteria are there after 5 hours?

C. A population of bacteria begins at 15,000. The population quintuples each hour. How many thousands of bacteria are there after 3 hours?

D. A bank account balance is $15. The account balance triples each year. What is the bank account balance, in dollars, after 5 years?

E. A bank account balance is $15,000. It grows by $3,000 each year. What is the bank account balance, in thousands of dollars, after 5 years?

5. Here are graphs of two exponential functions, f and g. If $f(x) = 100 \cdot \left(\frac{2}{3}\right)^x$ and $g(x) = 100 \cdot b^x$, which of the following could be the value of b?



A. $\frac{1}{3}$

B. $\frac{3}{4}$

C. 1

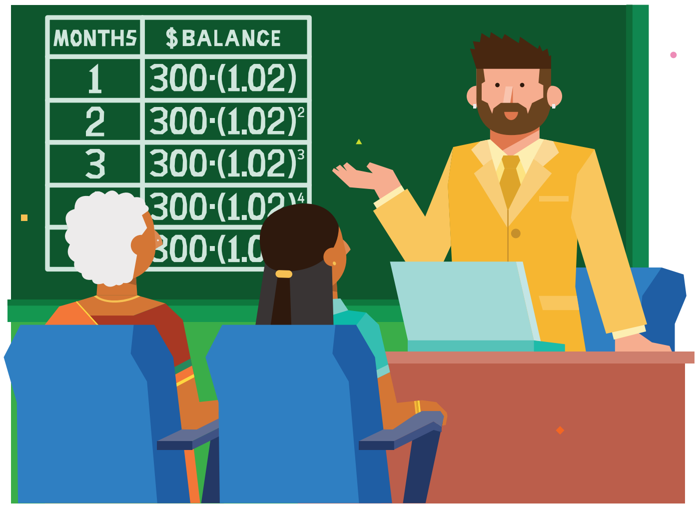
D. $\frac{3}{2}$

6. A person loans their friend $500. They agree to an annual interest rate of 5%. Write an expression for computing the amount owed on the loan, in dollars, after t years if no payments are made.

Unit 4 | Lesson 17

Compounding Interest

Let's explore different ways of repeatedly applying a percent increase.



Warm-up Five Years Later

Clare owes 12% interest each year on a $500 loan.

1. Write an expression to represent the balance after 5 years, if Clare makes no payments and takes no additional loans.

2. Evaluate your expression to determine the balance of the loan.

Log in to Amplify Math to complete this lesson online.

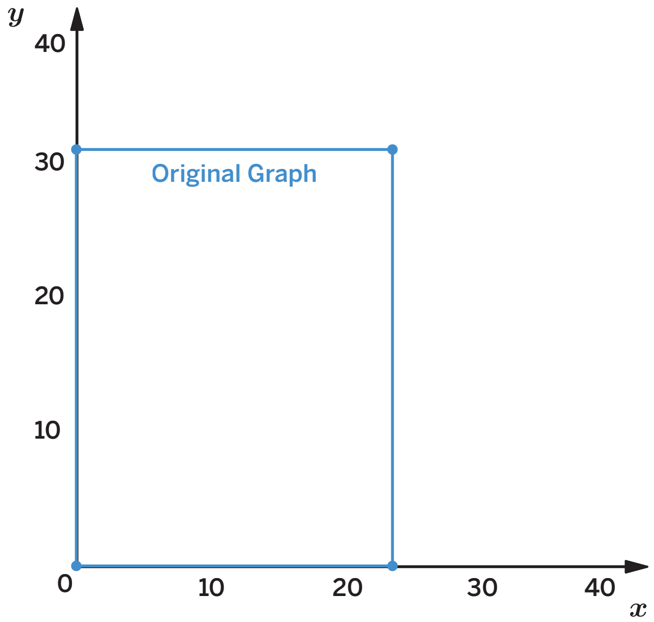
Name: ________ Date: ________ Period: ________

Activity 1 Resizing Images

Andre and Mai need to enlarge two images for a group project. The two images are the same size: 23.5 by 31 units.

Andre makes a scaled copy of his image, increasing the length and width by 10% each. It was still a little too small, so he increases them both by 10% again.

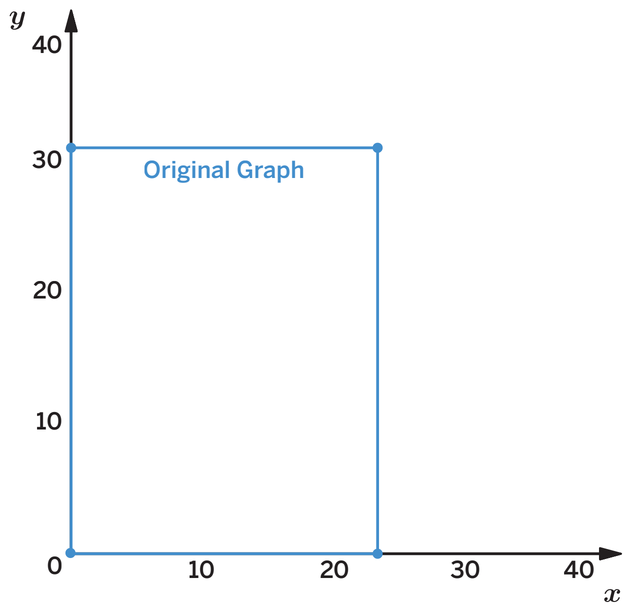
Sketch Andre's new image after performing Andre's actions.



Activity 1 Resizing Images (continued)

Mai says, "If I scale my image and increase the length and width by 20%, our images will be exactly the same size."

Sketch Mai's new rectangle after performing Mai's actions.



Do you agree with Mai? Explain or show your thinking.

Are you ready for more?

Adding 0.01 to a number 10 times is different from multiplying that number by 1.01 ten times, but the values are still quite close to one another. For example, $5 + (0.01) \cdot 10 = 5.5$, whereas $5 \cdot (1.01)^{10} \approx 5.523$. You could evaluate the first expression mentally, but you might want a calculator for the second. This introduces a neat mental math strategy.

a. The quantity $(1.02)^7$ is challenging to calculate by hand. Use mental math to compute $1 + (0.02) \cdot 7$ to determine a good approximation for it. Show the calculations you made.

b. Estimate $(0.99)^9$. Use a calculator to compare your estimate to the actual value. Show your calculations.

Name: ________ Date: ________ Period: ________

Activity 2 Earning Interest

In 2012, global health care giant GlaxoSmithKline agreed to pay a $\$3$ billion settlement to resolve allegations of unlawful promotion of prescription drugs, failing to report certain safety data, and false price reporting practices.

Let's explore how a $\$3$ billion settlement grows when placed in a savings account that has a monthly interest rate of 1%. Any earned interest is added to the account, and no other deposits or withdrawals are made.

1. What is the account balance after 6 months, 1 year, 2 years, and 5 years? Show your thinking.

2. Write an equation expressing the account balance a in terms of the number of months m.

3. How much interest will the account earn in 1 year? What percentage of the initial balance is that? Show your thinking.

4. The term *annual return* refers to the percent of interest an account holder could expect to receive in one year. If you represented the bank, would you advertise the account as having a 12% annual return? Why or why not? Explain your thinking.

STOP

Summary

In today's lesson . . .

You explored different ways of repeatedly applying a percent increase. Compounding happens when interest is calculated on money in a bank account or on a loan. An account that earns 2% interest every month *does not* actually earn 24% a year (which you might think, because $24 = 12 \cdot 2$). Suppose a savings account has $\$300$, and that no other deposits or withdrawals are made. The account balances after three months are shown in the table.

Number of months	Account balances ($)
1	$300 \cdot (1.02)$
2	$300 \cdot (1.02)^2$
3	$300 \cdot (1.02)^3$

$(1.02)^{12} \approx 1.2682$, so the account will grow by about 26.82% in one year. This rate is called the *effective interest rate.* It reflects how the account balance *actually* changes after one year.

If the account accrued 24% interest annually, this rate is called the *nominal interest rate.* It is the *stated rate of interest* and used to determine the amount for one year. A nominal interest rate can be used to determine a monthly, weekly, or daily rate.

Reflect:

Practice

Name: .... Date: .... Period: ....

1. Automobiles start losing value, or depreciating, as soon as they leave the car dealership. Five years ago, a family purchased a new car that cost $\$16{,}490$. If the car lost 13% of its value each year, what is the value of the car now?

2. The number of trees in a rainforest decreases each month by 0.5%. The forest currently has 2.5 billion trees. Write an expression to represent how many trees will be left in 10 years. Then, evaluate the expression.

3. From 2005 to 2015, a number of people p who were diagnosed with a newly mutated virus is modeled by the equation $p = 1500 \cdot (0.98)^t$, where t is the number of years since 2005.

   a. Based on the model, about how many people were diagnosed with this virus in 2005?

   b. Describe what is happening to the number of diagnoses each year between 2005 and 2015.

   c. About how many people were diagnosed with the virus in 2015? Show your thinking.

Practice

Name: Date: Period:

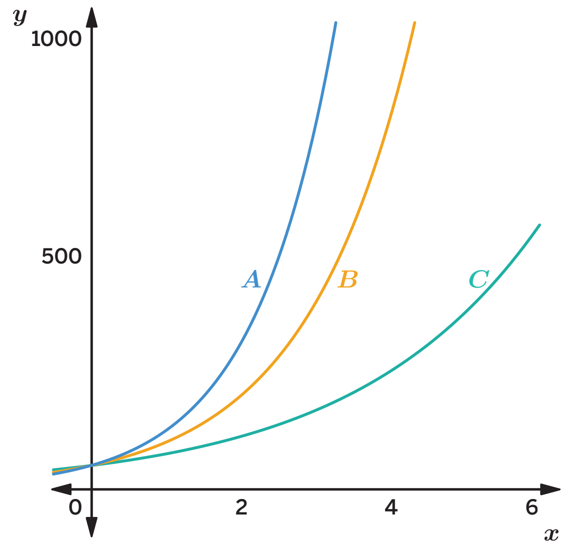
4. Here are the graphs of three equations.

$y = 50 \cdot (1.5)^x$

$y = 50 \cdot (2)^x$

$y = 50 \cdot (2.5)^x$

Which equation corresponds with each graph? Explain your thinking.



5. A major retailer has a staff of 6,400 employees for the holidays. After the holidays, it will decrease its staff by 30%. How many employees will it have after the holidays?

6. Simplify each expression by rewriting each as an expression with one power.

a. $(1.12^5)^5$

b. $\frac{1.12^{30}}{1.12^5}$

c. $(1.12 \cdot 1.12)^{100}$

d. $(1.12)^2 \cdot (1.12 \cdot 1.12)^{100}$

Name: ______________________ Date: __________ Period: __________

Unit 4 | Lesson 18

Expressing Exponentials in Different Ways

Let's write exponential functions in different, yet equivalent ways.

Warm-up Math Talk

Decide if each expression is equal to $(1.21)^{100}$, and explain your thinking.

Expression 1: $((1.21)^{10})^{10}$	Expression 2: $((1.21)^{50})^{50}$	Expression 3: $((1.1)^{2})^{100}$	Expression 4: $(1.1)^{200}$

Discussion Supports:
Can you represent your thinking algebraically so another student can understand your work without needing a written explanation?

Log in to Amplify Math to complete this lesson online.

Activity 1 Population Projections

From 1790 to 1860, the United States population, in thousands, could be modeled by the equation $P = 4000 \cdot (1.031)^t$ where t is the number of years since 1790.

1. About how many people were living in the U.S. in 1790? What about in 1860? Show your thinking.

2. What was the approximate annual percent increase predicted by the model?

3. In 2017, the U.S. population was estimated at $326{,}600{,}000$. What does the model predict for the population in 2017? Is it accurate? Explain your thinking.

4. What percent increase does the model predict over the course of a decade (10 years)? Explain your thinking.

5. Suppose d represents the number of decades since 1790. Write an equation that models P, the U.S. population (in thousands), in terms of d.

6. What percent increase does the model predict over the course of a century (100 years)? Explain your thinking.

7. Suppose c represents the number of centuries since 1790. Write an equation that models P, the U.S. population (in thousands) in terms of c .

Name: ______ Date: ______ Period: ______

Activity 2 Simulating an Epidemic

Medical researchers use mathematical models to better understand how people can stop the spread of disease, and to predict how many people will become infected.

In this activity, you will simulate exponential growth by rolling a die. Use the rules for each scenario to complete its table.

Scenario 1: 0% vaccination rate
At first, just 1 person is infected. For each round, roll a die, and record the number rolled. The total number of infected people after that round will be the number of infected people from the previous round multiplied by your roll.

For example, if there were 10 infected people in the previous round, and you roll a 3, there will now be 30 infected people.

Round	Number rolled	Total infected
0	—	1
1		
2		
3		
4		
5		

Scenario 2: 50% vaccination rate
Now, half of the population has been vaccinated and cannot become infected. To determine the number of newly infected people each round, multiply the number of infected people from the previous round by *half* of the number you rolled with the die. Use the same sequence of rolls from Scenario 1 (i.e., do not roll the die again). Round your answers to the nearest whole number, if necessary.

Round	Number rolled	Total infected
0	—	1
1		
2		
3		
4		
5		

In both scenarios, the growth of the disease depended on what you rolled. With a fair die, the average roll will be 3.5 (the average of the numbers from 1 through 6). If you halve the roll, the average result would be half of 3.5, or 1.75.

Activity 2 Simulating an Epidemic (continued)

1. Write a function for T_1, the total number of infected people in Scenario 1, as a function of the round n, assuming a growth factor of 3.5 and 1 person being infected in Round 0.

2. Write a function for T_2 the total number of infected people in Scenario 2, as a function of the round n, assuming a growth factor of 1.75 and 1 person being infected in Round 0.

3. What are the growth rates in the two scenarios? Express your rates as percentages.

4. Using your functions, can you find any times when the number of infected people in the two scenarios would be similar? How many rounds in Scenario 1 and how many rounds in Scenario 2 would result in a similar number of people becoming infected?

STOP

Name: ________ Date: ________ Period: ________

Summary

In today's lesson . . .

You explored how to write exponential expressions and functions in different ways. Different ways of writing expressions and functions helps to highlight different aspects of a situation or to better understand the situation. You have previously seen exponential expressions written with their growth factor. You can also describe the growth rate of an exponential situation, which is the percent change. In any situation involving percent change, it is important to note if the change is an increase or a decrease.

The table illustrates the connection between *growth factors* and *growth rates*.

Percent increase:
An amount increases by 20% each year.

Growth factor	Growth rate
1.20 or $1 + 0.20$	20%
The amount is multiplied by a factor of 1.03 each year.	The amount grows (increases) at a rate of 3% per year.

Percent decrease:
An amount decreases by 20% each year.

Growth factor	Growth rate
0.80 or $1 - 0.20$	-20%
The amount is multiplied by a factor of 0.80 each year.	The amount decays (decreases) at a rate of 3% per year.

Reflect:

Practice

Name: .... Date: .... Period: ....

1. For each growth rate, find the associated growth factor.

a. 30% increase

b. 30% decrease

c. 2% increase

d. 2% decrease

e. 0.04% increase

f. 0.04% decrease

2. In 1990, the population of India was about 870.5 million. By 1995, there were about 960.9 million people. The equation $p = 870.5 \cdot (1.021)^t$ approximates the number of people, in millions, in terms of the number of years t since 1990.

a. By what factor does the number of people grow in one year?

b. Write an equation that gives the number of millions of people p in terms of the number of decades d since 1990.

c. Use the model $p = 870.5 \cdot (1.021)^t$ to predict the number of people in India in 2015.

d. In 2015, the population of India was 1,311 million. How does this compare with the predicted number?

3. An investor paid \$156,000 for a condominium in Texas in 2008. The value of the homes in the neighborhood have been appreciating (i.e., increasing) by about 12% annually. Select *all* the expressions that could be used to calculate the value of the house, in dollars, after t years.

A. $156000 \cdot (0.12)^t$

B. $156000 \cdot (1.12)^t$

C. $156000 \cdot (1 + 0.12)^t$

D. $156000 \cdot (1 - 0.12)^t$

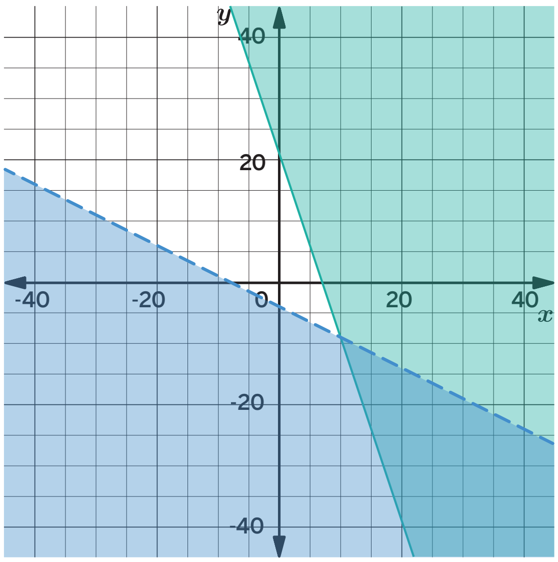
E. $156000 \cdot \left(1 + \frac{0.12}{12}\right)^t$

Name: .... Date: .... Period: ....

Practice

4. Han loans Andre \$300. Andre agrees to repay Han at an annual interest rate of 5%. Write an expression for computing the amount owed on the loan, in dollars, after t years if no payments are made.

5. Two inequalities are graphed on the same coordinate plane. Select *all* points that are solutions to the system of the two inequalities.



A. $(0, 10)$

B. $(0, -10)$

C. $(10, 0)$

D. $(-10, 0)$

E. $(10, -10)$

F. $(-10, -10)$

G. $(14, -12)$

H. $(20, -20)$

6. Determine the value of the following expressions:

a $10 \cdot (1 + 0.05)$

b $10 \cdot \left(1 + \frac{0.05}{2}\right)^2$

c $10 \cdot \left(1 + \frac{0.05}{12}\right)^{12}$

Unit 4 | Lesson 19

Credit Cards and Exponential Expressions

Let's find out what happens when we repeatedly apply the same percent increase at different intervals of time.

Warm-up Returns Over Three Years

A bank account has an initial balance of $\$1,000$ and earns 1% monthly interest. Each month, the interest is added to the account and no other deposits or withdrawals are made.

To calculate the account balance in dollars after 3 years, Elena wrote the expression $1000 \cdot (1.01)^{36}$ and Tyler wrote the expression $1000 \cdot [(1.01)^{12}]^3$.

Solve the following problems, and then discuss with a partner.

1. Why do Elena's expression and Tyler's expression both represent the account balance correctly?

2. Kiran said, "The account balance is about $1000 \cdot (1.1268)^3$." Do you agree? Why or why not?

Log in to Amplify Math to complete this lesson online.

Name: ________________ Date: ________ Period: ________

Activity 1 Interest Calculations

The chief operating officer (COO) of MedFund, a start-up company, plans to deposit $\$1,000$ in an interest-bearing bank account. The bank provides two options: one account with 7% annual interest, and another account that provides half the interest (3.5%) but compounds twice as often (semi-annually, or every six months). Both accounts are advertised as having a *nominal* interest rate of 7%.

1. What do you think it means that both accounts have a *nominal* interest rate of 7%?

The COO wants to deposit the company's money in the first account, reasoning the account would then have $\$1,070$ next year.

2. The company's chief financial officer (CFO) thinks the company should place their $\$1,000$ in the second account. What would the balance be after one year for this second account?

3. While this second account also had a nominal interest rate of 7%, what do you think was its *effective* interest rate?

Activity 1 Interest Calculations (continued)

The CFO looks into more options for earning interest over the next 6 years.

Here are three expressions and three descriptions. In each case, $1,000 has been deposited in an interest-bearing bank account. No withdrawals or other deposits (aside from the earned interest) are made for 6 years.

Expressions	Descriptions
A. $1000 \cdot \left(1 + \frac{0.07}{12}\right)^{72}$	**1.** 7% nominal annual interest, compounded twice each year.
B. $1000 \cdot \left(1 + \frac{0.07}{2}\right)^{12}$	**2.** 7% nominal annual interest, compounded monthly.
C. $1000 \cdot \left[\left(1 + \frac{0.07}{12}\right)^{12}\right]^{6}$	**3.** 7% nominal annual interest, compounded every two months.

4. Match the expressions with their descriptions. One description will not have a matching expression.

5. For the description without a match, write an expression that matches it.

6. Descriptions 1, 2, and 3 all have nominal interest rates of 7%. Which of the three has the highest *effective* interest rate? Explain your thinking.

Name: ________ Date: ________ Period: ________

Activity 2 Misleading Credit Card Rates

A credit card company lists a nominal annual percentage rate (APR) of 24%, but compounds interest monthly at 2% per month. In other words, for every month you do not pay your credit card bill, the credit card company will charge you 2% interest on what you owe.

Suppose you spend $1,000 using your credit card, and you make no payments or other purchases. Assume the credit card company does not charge any fees other than the interest.

1. Write expressions for the amount you would owe after 1 month, 2 months, 6 months, and a year (12 months).

2. Write an expression for the amount you would owe, in dollars, after m months without payment.

3. How much would you owe after 1 year without payment? What is the *effective* APR of this credit card?

4. Write an expression for the amount you would owe in dollars, after t years without payment. Be prepared to explain your expression.

Are you ready for more?

A bank account has an initial balance of $800 and accrues a nominal annual interest of 12%. Any earned interest is added to the account, but no other deposits or withdrawals are made. Write an expression that represents the balance for each of the following.

1. After 5 years, if interest is compounded n times per year.

2. After t years, if interest is compounded n times per year.

3. After t years, with an initial deposit of P dollars and an annual interest percentage rate of r, compounded n times per year.

Activity 3 Which Would You Choose?

Saving money through bank accounts or retirement accounts like 401(k)s is a common way to build wealth. Mathematicians and economists, like Sepideh Modrek, study how different people save. Suppose you have $500 to invest and can choose between two investment options.

Option 1: Every 3 months, 3% interest is applied to the balance.
Option 2: Every 4 months, 4% interest is applied to the balance.

Choose *one* of the options, and build a mathematical model for the chosen investment option. Then use your model to support your investment decision. Remember to state your assumptions about the option.

Stronger and Clearer: Share your model, assumptions, and chosen investment option with another group to both give and receive feedback. Use the feedback you receive to revise your response.

Featured Mathematician

Sepideh Modrek

An assistant professor of Economics at San Francisco State University's Health Equity Institute (HEI), Sepideh Modrek studies the effects of employment security and population-based policies on health, and how political uncertainty can affect behavior. With Kai Yuan Kuan and Mark R. Cullen, she authored a paper that analyzed ethnic and racial disparities in savings behavior, specifically in retirement accounts. They found that the avoidance of risk (or unreliability, which accounts like 401(k)s have) partially explained the observed differences.

Name: .... Date: .... Period: ....

Activity 4 Changes Over the Years

A local university offers programs designed to provide educational opportunities for students in the field of public health and infectious diseases. The function $f(x) = 15 \cdot (1.07)^x$ models the cost of tuition, in thousands of dollars, for one of the programs at the university, x years since 2017.

1. What is the cost of tuition at this university in 2017?

2. At what annual percentage rate does the tuition increase?

3. Assume that before 2017, the tuition had also been growing at the same rate as it did after 2017. What was the tuition in 2000? Show your thinking.

4. What was the tuition in 2010?

5. Assuming this rate, what will the tuition be when you graduate from high school?

STOP

Summary

In today's lesson . . .

You explored situations in which a percent increase is applied repeatedly over different time intervals. The overall amount of interest earned (or owed) on a balance is affected by how often that interest is calculated and added to the previous amounts, or *compounded* over time. *Compound interest* refers to the interest that is calculated on the initial amount plus any previously earned interest, and is often calculated more than once a year. More frequent compounding means the amount will grow more quickly.

Suppose a bank account has a balance of $\$1,000$ and a *nominal annual interest rate* of 6% per year, compounded over different time intervals. The following table shows the corresponding expressions for each compounding interval, as well as the account balance after one year.

Compounding interval	Number of times compounded per year	Account balance after one year ($)
annually (12 months)	1	$1000 \cdot (1 + 0.06) = 1060$
twice a year (6 months)	2	$1000 \cdot (1 + 0.03)^2 = 1060.90$
quarterly (3 months)	4	$1000 \cdot (1 + 0.015)^4 \approx 1061.36$
monthly (1 month)	12	$1000 \cdot (1 + 0.005)^{12} \approx 1061.68$

Reflect:

Name: .... Date: .... Period: ....

Practice

1. The population of a city in 2020 is $50,000$, and it grows by 5% each year.

   a. Write a function that models the population $f(t)$ of the city t years after 2020.

   b. What is the predicted population in 2027?

   c. What is the predicted population of the city in 2030? In 2040?

   d. By what factor is the population predicted to grow between 2020 and 2030? Between 2030 and 2040?

2. A person charges $\$100$ to a credit card with a 24% nominal annual interest rate. Assuming no other charges or payments are made, determine the balance on the card, in dollars, after one year if interest is calculated:

   a. Annually

   b. Every 6 months

   c. Every 3 months

   d. Monthly

   e. Daily

3. A couple has $\$5,000$ to invest and is choosing between three investment options.

   Option A: $2\frac{1}{4}\%$ nominal annual interest, compounded quarterly.

   Option B: 3% nominal annual interest, compounded every 4 months.

   Option C: $4\frac{1}{2}\%$ nominal annual interest, compounded semi-annually.

   If they make no deposits and no withdrawals for 5 years, which option will give them the largest balance after 5 years? Use a mathematical model for each option to explain your thinking.

Practice

Name: ______ Date: ______ Period: ______

4. At the end of each year, 10% interest is charged on a $\$500$ loan. The interest applies to any unpaid balance on the loan, including previous interest. Select *all* the expressions that represent the loan balance after two years, if no payments are made.

A. $500 + 2 \cdot (0.1) \cdot 500$

B. $500 \cdot (1.1) \cdot (1.1)$

C. $500 + (0.1) + (0.1)$

D. $500 \cdot (1.1)^2$

E. $(500 + 50) \cdot (1.1)$

5. Suppose m and c each represent the position number of a letter in the alphabet, but m represents the letters in the original message, and c represents the letters in a secret code. The equation $c = m + 7$ is used to encode a message. Assume the alphabet loops. For example, counting three back from the letter A would be the letter *X*.

a Write an equation that can be used to decode the secret code into the original message.

b What does this code say: "AOPZ PZ AYPJRF!"?

6. Complete the table to order the functions in descending order of steepness.

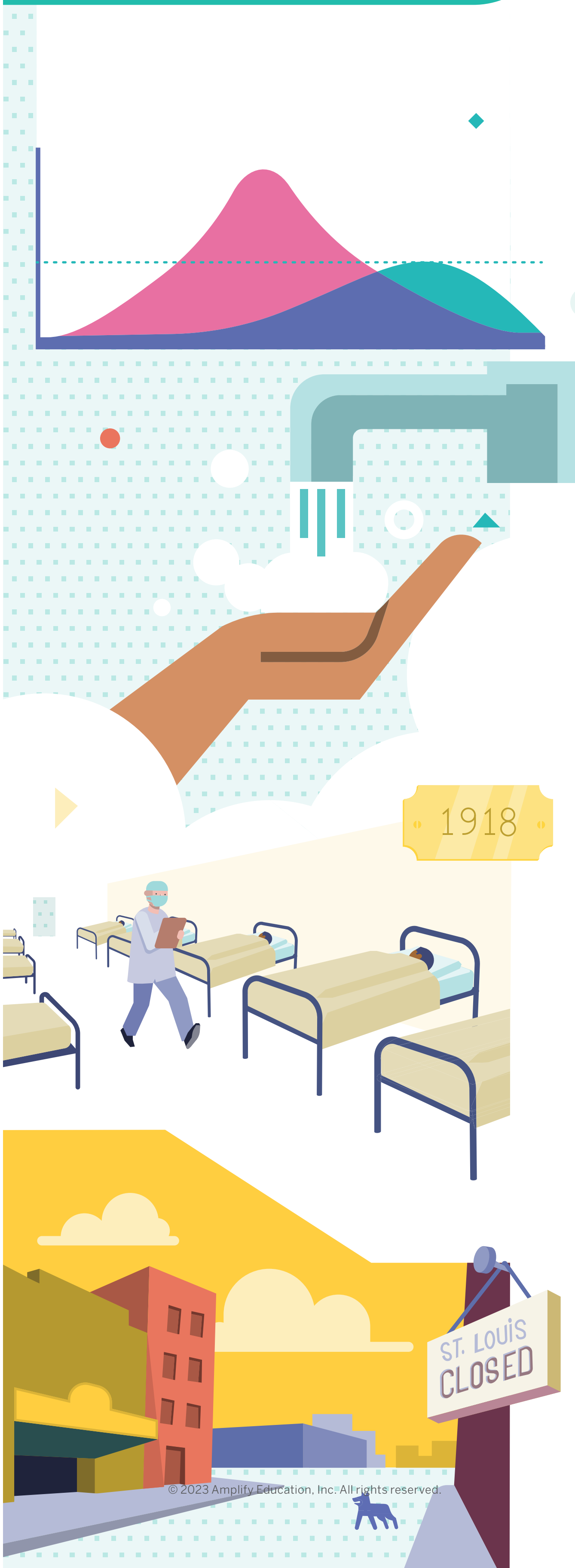
$f(x) = \frac{1}{3}x$ $\quad g(x) = 3$ $\quad h(x) = 3x$ $\quad j(x) = \frac{7}{2}x$ $\quad k(x) = 3.25x$

	Most steep
	Least steep

SUB-UNIT

5 Comparing Linear and Exponential Functions

Narrative Connections



How does distance make the curve grow flatter?

In the winter of 2019, at the start of the COVID-19 pandemic, the world was introduced to the phrase "flattening the curve."

The idea was that if enough people practiced good hygiene and socially distanced, a country could *slow down* a disease and keep it from overwhelming their hospitals.

But how do we know it can work? Here's a tale of two cities:

In 1918, there was an outbreak of a deadly flu virus in the U.S. Despite warnings from medical experts, the city of Philadelphia decided to continue with a parade that brought out hundreds of thousands of its citizens. Over the next six months, 16,000 residents lost their lives.

Meanwhile, in St. Louis, the story was quite different. After its first reported cases, the city went into lockdown, closing schools, movie theaters, churches, and other public gatherings. By the end of the outbreak, St. Louis had one-eighth of the fatalities that Philadelphia saw.

As telling as this tale may be, graphs tell the story even better. Mathematical representations like graphs tell us not only how fast a disease is spreading, but also what strategies we can use to slow it down and save lives.

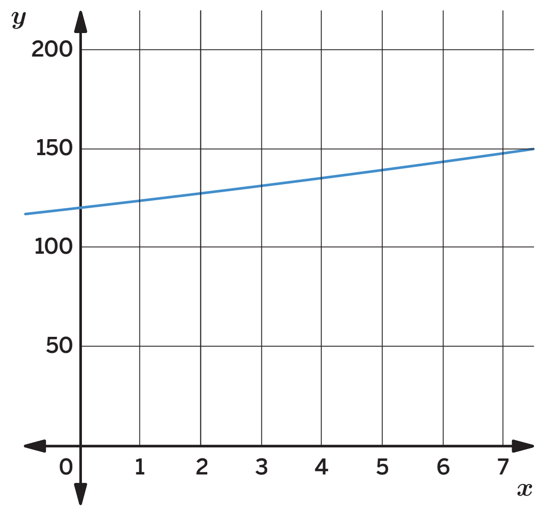
Unit 4 | Lesson 20

Which One Changes Faster?

Let's compare linear and exponential functions as they increase.

Warm-up What Is the Function?

Refer to the graph.



1. Which of the following equations do you think the graph represents? Use the graph to support your thinking.

$y = 120 + (3.7) \cdot x$

$y = 120 \cdot (1.03)^x$

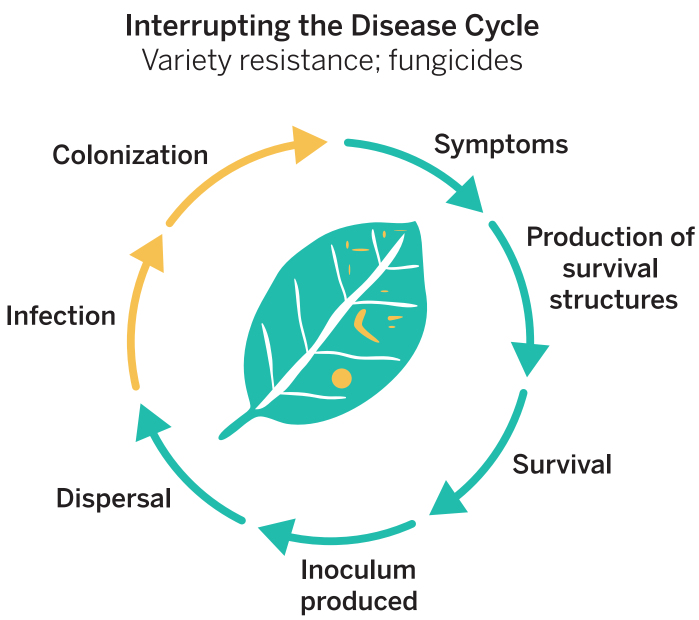
2. What information might help you decide more easily whether the graph represents a linear or an exponential function?

Log in to Amplify Math to complete this lesson online.

Name: .... Date: .... Period: ....

Activity 1 Plant Disease

Did you know plants get sick too? It is estimated that plant diseases kill off nearly half the food produced in the world, costing farmers worldwide hundreds of billions of dollars every year.



Many plant diseases are caused by bacteria that gain entry into the plant and reproduce inside the leaves, stems, and roots. Then, the bacteria spread to other plants through the air, or with the help of animals who unwittingly eat the diseased crops.

Beans can become infected with a bacterium that causes halo blight on their leaves, resulting in discoloration or abnormally small beans. Flaxseed, another edible plant, is susceptible to Fusarium wilt, a soil-borne disease that causes root rot.

A local farmer is modeling the spread of plant disease on her farm. There are 500 flaxseed plants infected with Fusarium wilt. In her model, this amount increases by 4% of the original 500 each day. There are also 450 bean plants infected with halo blight. In her model, 4% more bean plants are infected each day than the day before.

Flaxseed: Fusarium wilt

Days	Number of plants infected
0	500
1	520
2	540

Beans: Halo blight

Days	Number of plants infected
0	450
1	468
2	487

Activity 1 Plant Disease (continued)

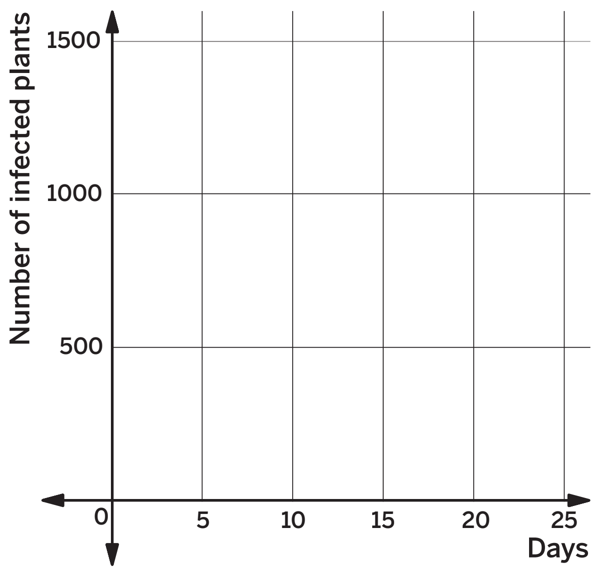
1. Describe how the number of infected flaxseed plants is changing.

2. In the bean plant model, how were 468 and 487 calculated?

3. For each plant species, write an equation to represent the relationship between the number of plants infected and the number of days.

4. Which infected plants should the farmer treat first? Use your equations or calculations to support your answer.

5. Use graphing technology to graph the two disease scenarios and show how quickly the disease spreads for each plant.



Name: ________ Date: ________ Period: ________

Activity 2 Reaching 2,000

Consider the functions $f(x) = 2x$ and $g(x) = (1.01)^x$.

1. Complete the table of values for each function.

x	$f(x)$	$g(x)$
1		
10		
50		
100		
500		

2. Based on the table of values, which function do you think grows faster? Explain your thinking.

3. Which function do you think will reach a value of 2,000 first? Show your thinking.

Are you ready for more?

Consider the functions $g(x) = x^5$ and $f(x) = 5^x$. While it is true that $f(7) > g(7)$, this fact is challenging to check using mental math. Determine a value of x for which properties of exponents allow you to conclude that $f(x) > g(x)$ without a calculator.

STOP

Summary

In today's lesson . . .

You looked at different examples of exponential functions and linear functions. You studied their graphs, equations, and tables of values.

Sometimes, an exponential function might seem to grow too slowly to catch up to a linear function with a large y-intercept or slope. But at some value of x down the line, the exponential function will *always* catch up to and surpass the linear function.

When determining whether or when one function will overtake another, you can use a table, substitute to evaluate the function at large values, or use a graph with large axes limits to examine how the functions grow over time.

Reflect:

Practice

Name: ______ Date: ______ Period: ______

1. The functions a, b, c, d, e, and f are given. Classify each function as *linear*, *exponential*, or *neither*.

a. $a(x) = 3x$ ______

b. $d(x) = 9 + 3x$ ______

c. $b(x) = 3^x$ ______

d. $e(x) = 9 \bullet 3^x$ ______

e. $c(x) = x^3$ ______

f. $f(x) = 9 \bullet 3x$ ______

2. *Graphing technology required.* Consider the functions $f(x) = 3x + 5$ and $g(x) = (1.1)^x$.

a. Complete the table with values of $f(x)$ and $g(x)$. When necessary, round to two decimal places.

b. Which function do you think grows faster? Explain your thinking.

x	$f(x)$	$g(x)$
1		
5		
10		
20		

c. Use graphing technology to create graphs representing $f(x)$ and $g(x)$. What axes limits should you use to see the value of x where $g(x)$ becomes greater than $f(x)$?

3. The functions m and n are given by $m(x) = (1.05)^x$ and $n(x) = \frac{5}{8}x$.

a. Which function reaches 30 first?

b. Which function reaches 100 first?

Practice

Name: .... Date: .... Period: ....

4. A line segment of length l is scaled by a factor of 1.5 to produce a segment with length m. The new segment is then scaled by a factor of 1.5 to give a segment of length n. What scale factor takes the segment of length l directly to the segment of length n? Explain your thinking.

5. Match each equation with its graph so that they represent the same function.

a $f(x) = |x|$ ....

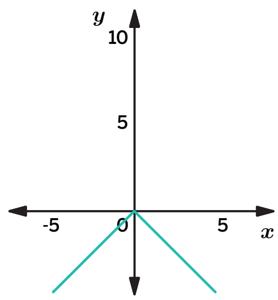
b $f(x) = -|x|$ ....

c $f(x) = |x+1|$ ....

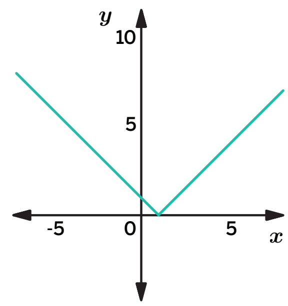
d $f(x) = |x-1|$ ....

e $f(x) = |x| + 1$ ....

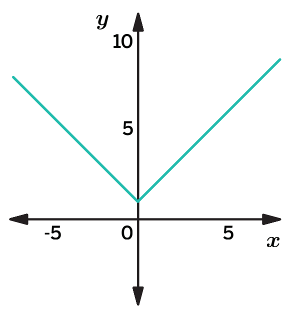
Graph A



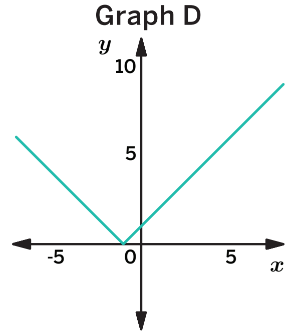
Graph B



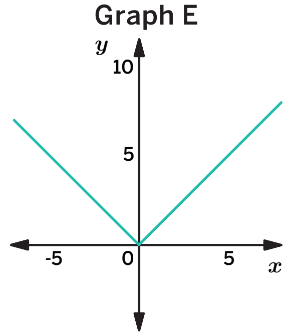
Graph C



Graph D



Graph E



6. For each expression, write an equivalent expression using as few terms as possible.

a $2(x+3)+1$

b $x^2 \bullet x^4 \bullet x^3$

c $\frac{b^{10}}{b^2}$

Name: ________ Date: ________ Period: ________

Unit 4 | Lesson 21

Changes Over Equal Intervals

Let's explore how linear and exponential functions change over equal intervals.

Warm-up Writing Equivalent Expressions

For each expression, write an equivalent expression using as few terms as possible.

1. $7p - 3 + 2(p + 1)$

2. $[4(n + 1) + 10] - 4(n + 1)$

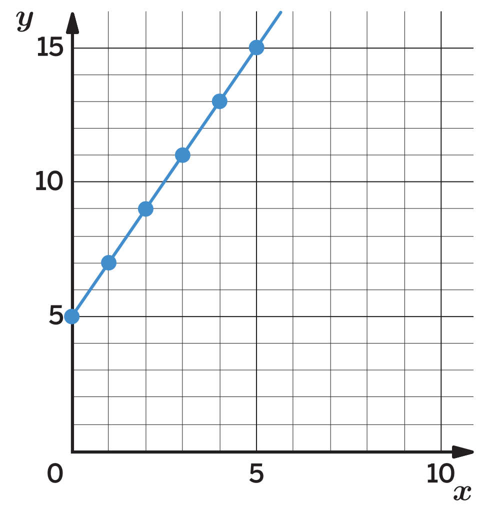
3. $9^5 \cdot 9^2 \cdot 9^x$

4. $\frac{2^{4n}}{2^n}$

Log in to Amplify Math to complete this lesson online.

Activity 1 Outputs of a Linear Function

Refer to the graph of $f(x) = 2x + 5$.



1. How do the values of $f(x)$ change whenever x increases by 1, such as when x increases from 1 to 2, or from 19 to 20? Explain your thinking.

2. Here is an expression you can use to determine the difference in the values of $f(x)$ when the input changes from x to $x + 1$.

   $[2(x + 1) + 5] - [2x + 5]$

   Does this expression have the same value as what you found in Problem 1? Explain your thinking.

3. How do the values of $f(x)$ change whenever x increases by 4? Explain your thinking.

4. Write an expression that shows the change in the value of $f(x)$ when the input value changes from x to $x + 4$.

5. Show or explain how your expression has a value of 8.

Name: ______ Date: ______ Period: ______

Activity 2 Outputs of an Exponential Function

The table shows several input and output values of the exponential function $g(x) = 3^x$.

x	$g(x)$
3	27
4	81
5	243
6	729
7	2187
8	6561
x	
$x + 1$	

1. How does $g(x)$ change every time x increases by 1? Show or explain your thinking.

2. Choose two new input values that are consecutive whole numbers and determine their output values. Record them in the empty rows of the table. How do the output values change for those two input values?

3. Complete the table of for x and $x + 1$.

4. Study the output values as x increases by 1. Do you still agree with your thinking in Problem 1? Show your thinking.

5. Choose two values of x where one is 3 more than the other (for example, 1 and 4). How do the output values of $g(x)$ change as x increases by 3? (Each group member should choose a different pair of numbers and study the outputs.)

Activity 2 Outputs of an Exponential Function (continued)

6. Write the expression for $g(x)$ when the input is x and $x+3$. Look at the change in the output as x increases by 3. Does it agree with your group's findings in Problem 5? Show or explain your thinking.

Are you ready for more?

For integer inputs, you can think of multiplication as repeated addition, and exponentiation as repeated multiplication:

$\mathbf{3 \cdot 4 = 3 + 3 + 3 + 3}$ **and** $\mathbf{3^4 = 3 \cdot 3 \cdot 3 \cdot 3}$

You could continue this process with a new operation called tetration. It uses the symbol ↑↑, and is defined as repeated exponentiation:

$\mathbf{3 \uparrow\uparrow 4 = 3^{3^{3^{3}}}}$

Compute $2 \uparrow\uparrow 3$ and $3 \uparrow\uparrow 2$.

If $f(x) = 3 \uparrow\uparrow x$, what is the relationship between $f(x)$ and $f(x+1)$?

Name: ________________ Date: ________ Period: ________

Activity 3 Price of Prescription Drugs

The journal, *Health Affairs*, reported that the price of brand-name oral prescription drugs rose by 9.2% per year between 2008 and 2016. The annual cost of injectable drugs rose by 15.1%. Consider two prescription drugs:

Medicine W is an oral drug. The price w of one dose of Medicine W is modeled by the function $w(t) = 10 \cdot (1.092)^t$, where t represents the number of years since 2008.

Medicine K is an injectable drug. The price k of one dose of Medicine K is modeled by the function $k(t) = 8 \cdot (1.151)^t$, where t represents the number of years since 2008.

1. What is the price of one dose of Medicine W in 2008? In 2016?

2. Which medicine had a higher price in 2008? Which medicine had a higher price in 2016?

3. Assuming these trends continue, use the price function of Medicine W to show that for any given year, $x + 1$, the price of Medicine W will have increased by 9.2% from the previous year, x.

4. If both medicines are equally effective options to treat a chronic disease that requires lifelong medication, which prescription drug would be the cheaper option for a patient who started taking the drug in 2008?

STOP

Summary

In today's lesson . . .

You explored how linear and exponential functions change over equal intervals. Linear and exponential functions each behave differently when their input values increase by the same amount. A *linear function* will always increase by the same amount whenever its input increases by 1. Meanwhile, an *exponential function* will always be multiplied by the same amount whenever its input increases by 1.

A linear function always increases (or decreases) by the same amount over equal intervals. An exponential function increases (or decreases) by equal factors over equal intervals.

Reflect:

Name: ________________ Date: ________ Period: ________

Practice

1. Whenever the input value of a function $f(x)$ increases by 1, the output value increases by 5. Which of these equations could define $f(x)$?

   A. $f(x) = 3x + 5$

   B. $f(x) = 5x + 3$

   C. $f(x) = 5^x$

   D. $f(x) = x^5$

2. The function $f(x)$ is defined by $f(x) = 2^x$. Which of the following statements is true about the values of $f(x)$? Select *all* that apply.

   A. When x increases by 1, $f(x)$ increases by 2.

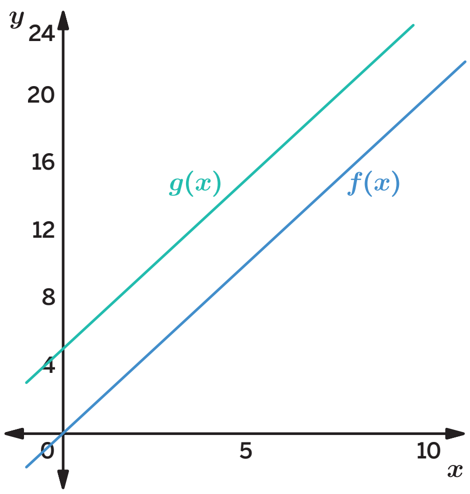
   B. When x increases by 3, $f(x)$ increases by 8.

   C. When x increases by 4, $f(x)$ increases by 4.

   D. When x increases by 1, $f(x)$ increases by a factor of 2.

   E. When x increases by 3, $f(x)$ increases by a factor of 8.

3. The two lines on the coordinate plane are graphs of functions $f(x)$ and $g(x)$.



   a. Use the graph to explain why the value of $f(x)$ increases by 2 each time x increases by 1.

   b. Use the graph to explain why the value of $g(x)$ increases by 2 each time x increases by 1.

Practice

Name: .... Date: .... Period: ....

4. For each of the functions $f(x)$, $g(x)$, and $h(x)$, the domain is $0 \leq x \leq 100$. For which function(s) is the average rate of change a good measure of how the function changes for this domain? Select *all* that apply.

   A. $f(x) = x + 2$

   B. $g(x) = 2^x$

   C. $h(x) = 111x - 23$

5. The average price of a gallon of regular gasoline in 2016 was $\$2.14$. In 2017, the average price was $\$2.42$ per gallon — an increase of 13%. At that rate, what will the average price of gasoline be in 2020?

6. The table shows the value of a new car that depreciates over time.

Time, t (years)	Value, $v(t)$ (thousands of \$)
0	35
1	30.8
2	26.8
3	23.6
4	20.9
5	18.4

   a. Can $v(t)$ be modeled accurately by a linear function? Explain your thinking.

   b. Can $v(t)$ be modeled accurately by an exponential function? Explain your thinking.

   c. Create a function to model the value v of the car, in thousands of dollars, in terms of the time t, in years.

Name: ______ Date: ______ Period: ______

Unit 4 | Lesson 22 – Capstone

COVID-19

Let's construct a linear or exponential model to represent the spread of COVID-19 and use our model to make predictions.

Warm-up A Little Distance

Refer to the image.

siam.pukkato/Shutterstock.com

1. What do you think will happen now that the first match is lit?

2. What would happen if the match sticks were placed farther apart?

Log in to Amplify Math to complete this lesson online.

Activity 1 COVID-19 Timeline

By March 2020, the novel coronavirus known as COVID-19 was beginning to spread in the U.S. The following tables show how many new confirmed cases there were per day, between March 2 and March 23, 2020.

Date	Number of new confirmed COVID-19 cases
March 2, 2020	16
March 3, 2020	21
March 4, 2020	36
March 5, 2020	67
March 6, 2020	83
March 7, 2020	117
March 8, 2020	119
March 9, 2020	201
March 10, 2020	270
March 11, 2020	245
March 12, 2020	405

Date	Number of new confirmed COVID-19 cases
March 13, 2020	556
March 14, 2020	674
March 15, 2020	702
March 16, 2020	907
March 17, 2020	1,399
March 18, 2020	2,444
March 19, 2020	4,043
March 20, 2020	5,619
March 21, 2020	6,516
March 22, 2020	8,545
March 23, 2020	10,432

1. Determine the average rate of change in the number of new cases for each time period.

a. March 2 to March 7

b. March 7 to March 12

c. March 13 to March 18

d. March 18 to March 23

Name: ______ Date: ______ Period: ______

Activity 1 COVID–19 Timeline (continued)

2. Which model — *linear*, *exponential*, or *neither* — is most appropriate for the spread of the virus? Explain your thinking.

3. Write a function that models the number of infections d days after March 2nd, 2020.

4. Use your function to predict the number of new infections on the following dates.

a April 2

b April 9

c April 16

d April 23

5. Do you think your function would accurately predict the number of infections over time? Explain your thinking.

Activity 2 Tracking the Spread

Throughout the COVID-19 pandemic, researchers like Carlos Rodriguez-Diaz have studied access to healthcare and medical treatment among different vulnerable populations.

The following table shows the number of new COVID-19 infections per 100,000 people in the U.S. in March, 2020, broken down by race and ethnicity.

	American Indian/Alaska Native	Asian/Pacific Islander	Black	Hispanic	White
March 7	1.23	1.45	2.18	1.46	1.44
March 14	3.98	4.73	8.33	5.68	5.80
March 21	9.00	13.76	28.49	21.22	13.05
March 28	12.87	19.69	41.52	34.95	16.51

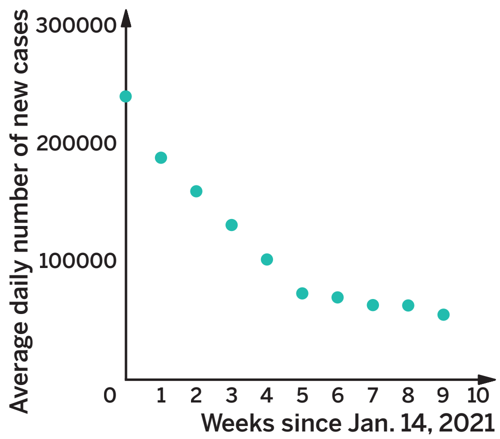
1. What do you notice? What do you wonder?

2. While the American Indian/Alaska Native population had fewer infections in early 2020, they had more infections per person than any other group between September, 2020 and February, 2021. Why do you think that might have happened?

Name: ________ Date: ________ Period: ________

Activity 3 Vaccine Response

By early 2021, multiple COVID-19 vaccines were being administered in several countries, including the U.S. Two vaccines that were most rapidly developed were RNA vaccines. When injected, the RNA in these vaccines enters cells and gets them to produce some of the viral proteins in a way that does not harm the body. The body then builds an immune response to these proteins, preparing it to fend off the real virus.



The graph shows the average number of new COVID-19 infections for 10 consecutive weeks in early 2021. During this time, vaccines were starting to be administered to the U.S. population.

1. Which model — *linear*, *exponential*, or *neither* — is most appropriate for the number of new cases in early 2021, as vaccines were being administered? Explain your thinking.

2. Write a function that models the number of daily infections w weeks after January 14th, 2021. Then sketch your function on the graph.

3. Do you think your function would accurately predict the number of infections over time? Explain your thinking.

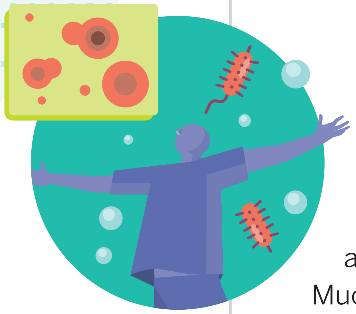
Featured Mathematician

Carlos Rodriguez-Diaz

A community health scientist, an activist, and an associate professor at George Washington University's Milken Institute School of Public Health, Carlos Rodriguez-Diaz has studied health inequities among vulnerable populations. He was the lead author on a 2020 paper examining the risk of COVID-19 infection and death among Latinos in the U.S. In addition to his academic work, Rodriguez-Diaz serves on the boards of multiple community-based organizations based in Puerto Rico and elsewhere in the U.S.

STOP

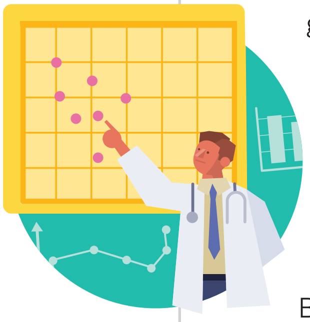
Unit Summary



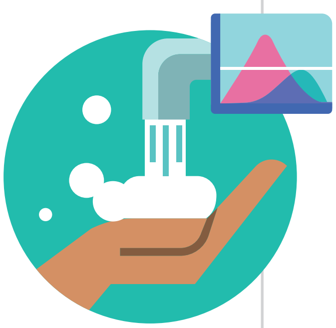
Diseases can be devastating. Beyond the damage they do to a person's health, contagious diseases spread from person to person, affecting entire communities, countries, and even the world.

The Black Plague of the 14th century killed approximately half of Europe's population. Much of America's indigenous population died from diseases brought by European explorers and settlers. One of these diseases, smallpox, ravaged Boston 300 years ago. And 100 years ago a strain of flu infected roughly a third of the world's population, killing approximately 50 million people.

As you saw in this unit, the way these diseases spread so quickly is a matter of mathematics. If one person infects two others, who each infect two others, and so on, the result is exponential growth. This means that a quantity is multiplied by a constant factor, the growth factor, over equal intervals.



Humanity again witnessed the exponential dangers of disease with the COVID-19 pandemic. In a matter of weeks, what started as a small number of infections suddenly exploded, overtaking entire communities and nations. But as with any tragedy, there remained hope. By understanding the mathematical nature of COVID-19's spread, we took measures to slow it down and stop it in its tracks. Social distancing, along with good hygiene and vaccination, can transform exponential growth into exponential decay.



As you probably know, there is more to the story. Exponential functions are good models, but infection rates can change over time, resulting in curves that are neither linear nor exponential. At the same time, populations have been affected differently by COVID's devastation. Some have been slower to receive assistance like vaccines. So while math can help you understand the mechanism and scope of tragedies, it can also help you make smart and equitable decisions on how to overcome them. This work is hard. It is also vitally important.

See you in Unit 5.

Name: ________________ Date: ________ Period: ________

Practice

The table gives the population of three cities over several decades. Refer to the table as you complete Problems 1–2.

City	1950	1960	1970	1980	1990	2000
Paris	6,300,000	7,400,000	8,200,000	8,700,000	9,300,000	9,700,000
Austin	132,000	187,000	254,000	346,000	466,000	657,000
Chicago	3,600,000	3,550,000	3,400,000	3,000,000	2,800,000	2,900,000

1. How would you describe the population change in each city?

2. What kind of model — *linear*, *exponential*, or *neither* — do you think is most appropriate for each city population?

Refer to Problem 2.

a Write an equation for each population that you think can be modeled by a linear or exponential function.

b Compare the graphs of your functions with the actual population data. How well do your models fit the data?

Practice

Name: ______ Date: ______ Period: ______

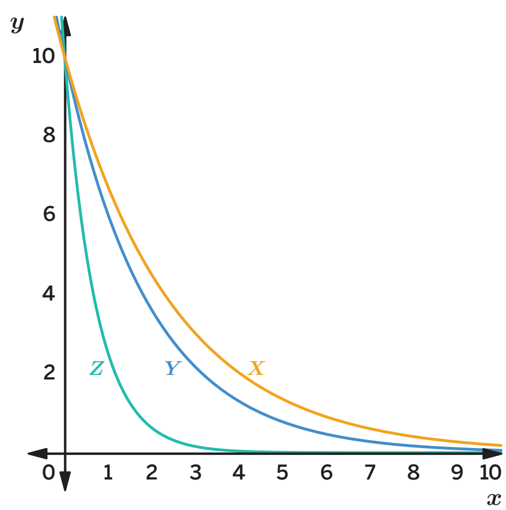
Refer to the table as you complete Problems 3 and 4.

Year	1804	1927	1960	1974	1987	1999	2007
World population, in billions	1	2	3	4	5	6	7

3. Would a linear or exponential function be more appropriate for modeling the world population growth over the last 200 years? Explain your thinking.

4. How many years did it take for the population to grow from 1 billion to 2 billion? From 6 billion to 7 billion? Why do you think that was that the case?

5. Which function in the graph decays the fastest? Which function decays the slowest?



My Notes:

UNIT 5

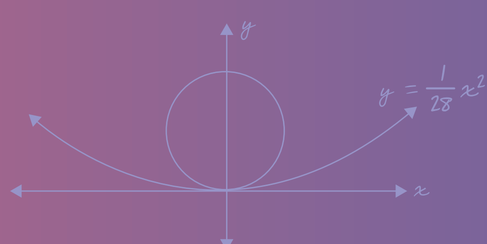
Introducing Quadratic Functions

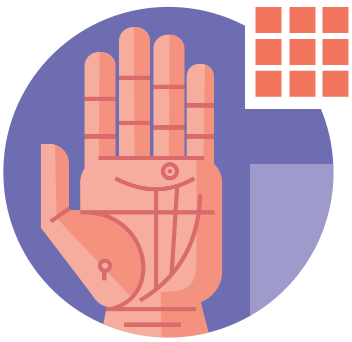
What goes up must come down. In this unit, you will study quadratic functions. By analyzing and comparing patterns, tables, graphs, and equations, you will gain an appreciation for the special features of quadratic functions and the situations they represent.

Essential Questions

- How are quadratic functions used to model, analyze, and interpret mathematical relationships?
- What characteristics of the graph of a quadratic function distinguish it from a linear function? An exponential function?
- What are the advantages of writing a quadratic function in vertex form? In standard form? In factored form?
- *(By the way, should Martians have to study quadratic functions?)*

What is the radius of the largest circle that fits snugly inside the parabola $y = \frac{1}{28}x^2$?





SUB-UNIT

1 A Different Kind of Change

Narrative: A square has more to offer than just four sides.

You'll learn . . .
- about another type of nonlinear change — quadratic change.

SUB-UNIT

2 Quadratic Functions

Narrative: From sports to freefall and earning revenue, quadratics model them all.

You'll learn . . .
- how quadratic functions model free-falling objects, projectile motion, and revenue.

SUB-UNIT

3 Quadratic Expressions

Narrative: Discover where the *quad-* in *quadratic* comes from.

You'll learn . . .
- about factored and standard forms of quadratic expressions.

SUB-UNIT

4 Features of Graphs of Quadratic Functions

Narrative: Discover the elegance and symmetry of the graph of a quadratic function.

You'll learn . . .
- about key features of the graphs of quadratic functions.
- about the vertex form of quadratic functions.

The Perfect Shot

Let's explore the mechanics of throwing a ball.

Warm-up Motion in Sports

Study the following images of athletes.

a

Leonard Zhukovsky/Shutterstock.com

b

Arturo Verea/Shutterstock.com

1. In your own words, describe how each athlete or their equipment is moving.

   a

   b

2. How is their motion similar? How is it different?

Log in to Amplify Math to complete this lesson online.

Name: ________ Date: ________ Period: ________

Activity 1 Foofoo's Buckets

Part A: Use the table to record notes on your partner's shooting technique. Place a check in the *Made it!* box if your partner makes any of their shots.

Position	Describe your partner's shooting technique *(What are they doing? How are they moving?)*
Front (Bucket 1)	☐ **Made it!**
Middle (Bucket 2)	☐ **Made it!**
Rear (Bucket 3)	☐ **Made it!**

Part B: With your partner, discuss the strategies you used when throwing the ball.

1. What was your strategy for making each bucket? (Include your partner's observations.)

2. Describe how the ball moved in the air after you threw it.

Compare and Connect: What similarities were there between your strategies? What differences were there?

Activity 2 Foofoo's Space Launch

Foofoo the Clown is adding a human cannonball act to his show and launching himself from a cannon. Foofoo is launched from three different cannons in three different positions.

1. What path will Foofoo take after he is launched from each cannon? Sketch what you think Foofoo's path will be in each diagram.

a


b


c

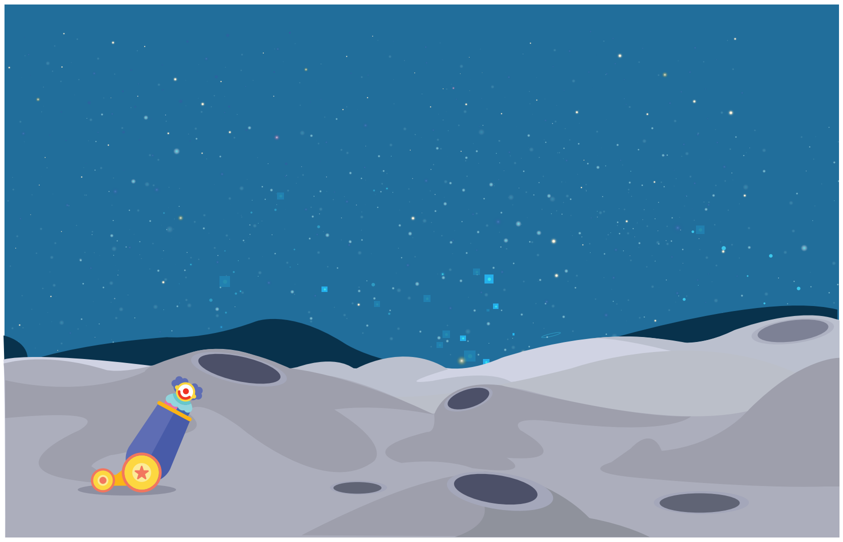

Name: ______________ Date: ______________ Period: ______________

Activity 2 Foofoo's Space Launch (continued)

2. How are the paths different? How are they similar?

What if Foofoo decided to make his act "out of this world"?

Foofoo's (hypothetical) launch from the Moon is shown.



3. What path do you think Foofoo will take after he is launched from the Moon?

4. The force of gravity is stronger on Earth than it is on the Moon. What effect do you think this force has on Foofoo? How does it change his path?

STOP

Narrative Connections

Unit 5 Introducing Quadratic Functions

Squares in Motion

They say, “What goes up must come down”— all thanks to gravity!

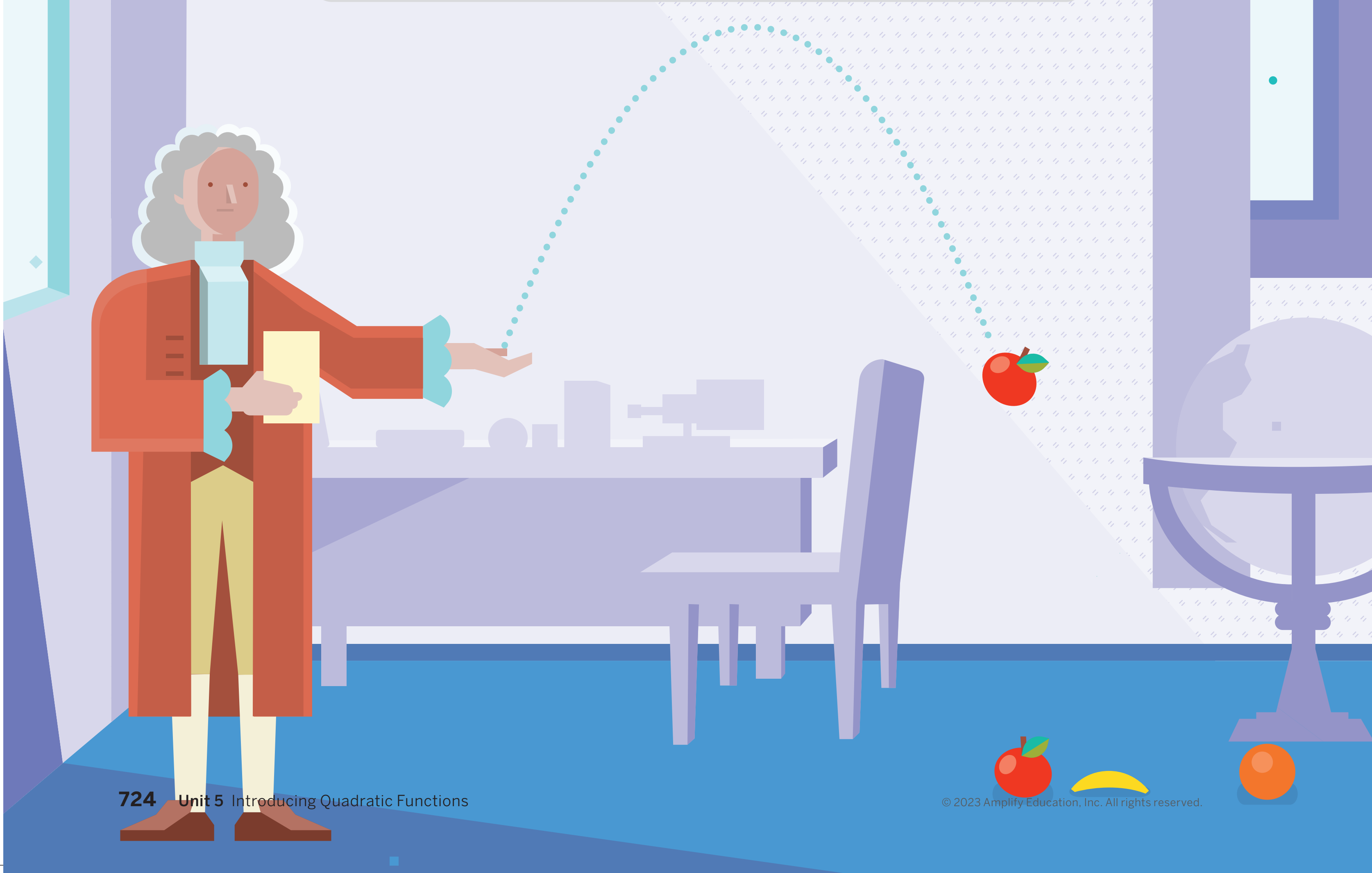
Even before its discovery by famed apple enthusiast Sir Isaac Newton, gravity has given life on Earth a certain panache.

You just saw it with your own eyes. Whether it was a gentle lob or your best LeBron James impression, the moment that ball leaves your hand, it makes a path through the air — not in a straight line, but as a curve.

But what *kind* of curve? Not like a circle, and not like an exponential curve either — but something else entirely . . .

In these next lessons, you will take a closer look at this kind of curve and the places it shows up, from a jump shot at the foul line to a dancer’s *grand jeté*.

Welcome to Unit 5.

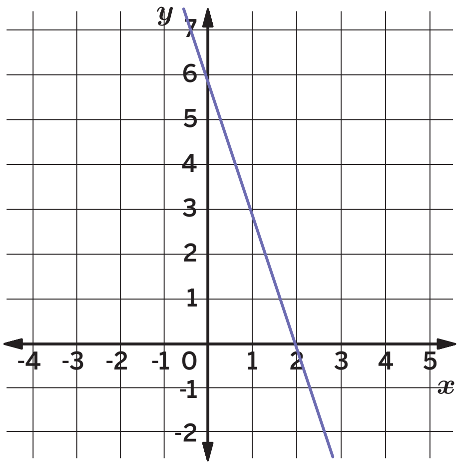


Practice

Name: .... Date: .... Period: ....

Calculators should not be used.

1. The graph represents the equation $y = -3x + 6$.

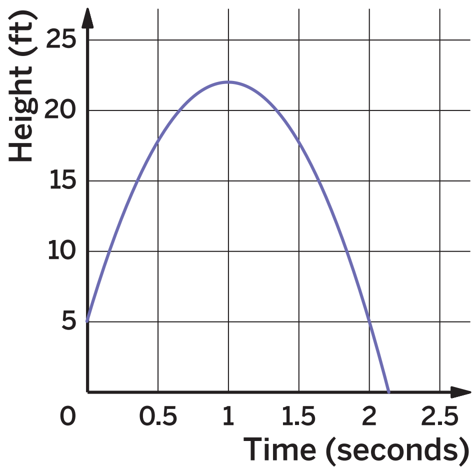


   a. Explain how you can use the equation to determine whether the point $(5, -9)$ is on the graph.

   b. What is the y-intercept of the graph? How is it related to the equation?

   c. What is the x-intercept of the graph? How is it related to the equation?

2. Andre is on a beach, and throws a rock up in the air so that it will land in the ocean. The graph shows the height of the rock, in feet, above the water as a function of time, in seconds.



   a. How high above the water was the rock when Andre threw it? Explain your thinking.

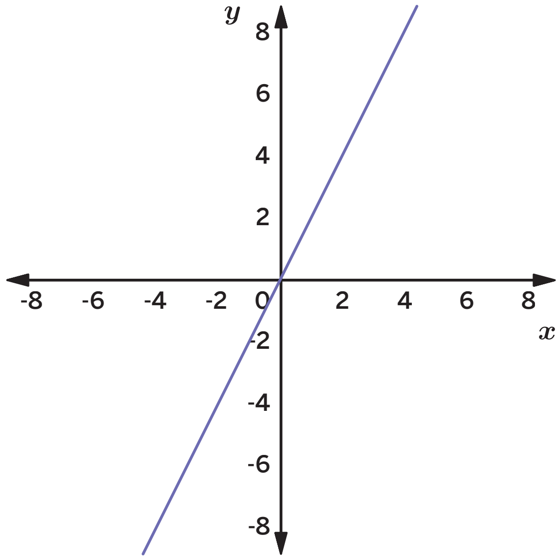
   b. When did the rock reach its maximum height? How high was it?

   c. Approximately when did the rock hit the ocean?

Practice

Name: .... Date: .... Period: ....

3. The graph of the equation $y = 2x$ is shown. How would the graph of the equation $y = 2x + 8$ look different from this graph?



A. It would have a steeper slope.

B. It would no longer be a line; it would curve upward.

C. It would be shifted 8 units to the right.

D. It would be shifted 8 units up.

4. A rectangle has a length of 6 cm and a width of $x + 10$ cm. Write expressions to represent the rectangle's perimeter and area.

Perimeter: **Area:**

SUB-UNIT

1 A Different Kind of Change

Narrative Connections

What's the best shape for a crystal ball?

There are many ways people try to predict the future: reading tea leaves, tarot cards, or palms, or gazing into crystal balls. But sometimes our best predictions come from simply paying attention.

That's what mathematicians do. By looking at patterns, we can predict how things will change.

Sometimes the patterns are straightforward; other times they're a bit more complex. You might even come across patterns within patterns — whose changes are governed by rules that seem to also be changing!

For the kinds of change we'll look at in this unit, we won't need a crystal ball. We can use something simpler. A shape we're all familiar with, one you've probably been drawing for almost as long as you've known how to hold a pencil. A shape whose features — its sides and area — speak to the hidden rules that guide how we play sports, move through spaces, or even how much we pay for things at a store.

Forget the crystal ball. We need a square.

A Different Kind of Change

Let's determine the rectangle with the greatest area.

Warm-up Notice and Wonder

Study the tables. What do you notice? What do you wonder?

Table A

x	y
1	0
2	5
3	10
4	15
5	20

Table B

x	y
1	3
2	6
3	12
4	24
5	48

Table C

x	y
1	8
2	11
3	10
4	5
5	-4

1. I notice . . .

2. I wonder . . .

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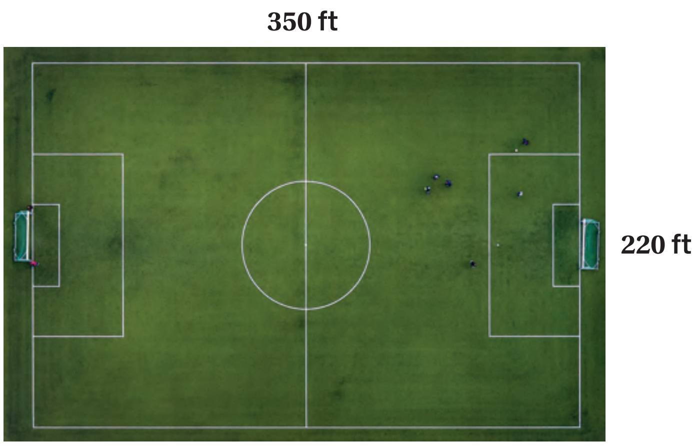
Name: ________ Date: ________ Period: ________

Activity 1 Constructing a Recreational Field

Noah is planning a new recreational field for his town. Local officials have asked Noah to make sure the area is rectangular and enclosed with exactly 1,000 feet of fencing.

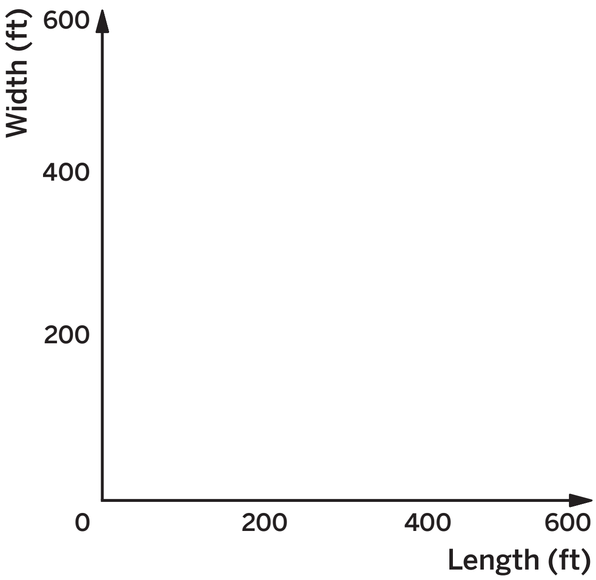
A standard-size soccer field



AndrewStarikov/Shutterstock.comShutterstock.com

1. Determine the perimeter and area of the standard-size soccer field shown.

2. Draw some possible diagrams of Noah's recreational field. Label the length and width of each rectangle.



Activity 1 Constructing a Recreational Field (continued)

3. Use the table to organize the different length and width combinations of the field. Determine the perimeter and area of each field.

Length (ft)	Width (ft)	Perimeter (ft)	Area (ft^2)

4. What length and width of the field produces the greatest possible area? Explain or show your thinking.

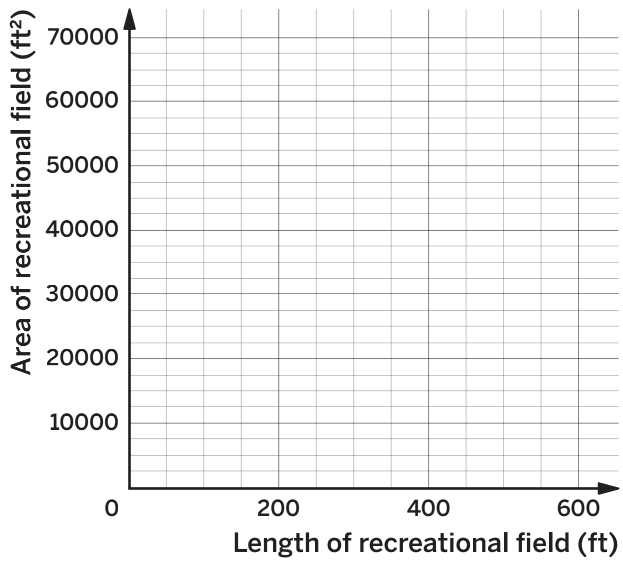
Discussion Support: During the class discussion, listen carefully to the observations your classmates share. Using appropriate math language, restate what you hear in your own words.

Name: ____________ Date: ____________ Period: ____________

Activity 2 Plotting the Measurements of the Recreational Field

1. Plot the values from your table in Activity 1 for the length and area of the recreational field on the coordinate plane.

2. What do you notice about the plotted points?



3. The points $(150, 52500)$ and $(350, 52500)$ represent possible lengths and areas of the recreational field. Plot these two points on the coordinate plane, if you have not already done so. What do these points represent in this scenario?

4. Does the point $(225, 56250)$ represent a possible length and area of the recreational field? Explain or show your thinking.

5. **a** What coordinates correspond to the maximum area of the field?

b What are the dimensions of the field for this area?

c What is the shape of the field? Explain your thinking.

Activity 3 The Length and Width

1. A rectangle is 11 m long and 14 m wide. Sketch this rectangle and determine its area.

2. A second rectangle is 14 m long and 11 m wide. Sketch this rectangle and determine its area.

3. What happens to the area of a rectangle when you interchange its length and width? Explain or show your thinking.

4. Plot three more points on your graph in Activity 2. What patterns would you notice if you were to plot more length and area pairs on the graph?

STOP

Name: ______ Date: ______ Period: ______

Summary

In today's lesson . . .

You explored the relationship between the side lengths and the area of a rectangle when the perimeter did not change.

For a rectangle with a fixed perimeter, the relationship between the length and the width is linear. As the length increases, the width decreases, and vice versa.

The length and area of a rectangle with a fixed perimeter have a different relationship. As the length of the rectangle increases, the area increases up to a point, and then decreases.

This relationship is not linear. It is a new relationship you will be exploring further in this unit.

Reflect:

Practice

Name: ______ Date: ______ Period: ______

1. Here are a few pairs of positive numbers whose sum is 50.

First number	Second number	Product
1	49	
2	48	
10	40	

a. Calculate the product of each pair of numbers.

b. Determine a pair of positive numbers that have a sum of 50 and will produce the greatest possible product.

c. Explain how you determined which pair of numbers have the greatest product.

2. The table shows some possible lengths and widths of a rectangle whose perimeter is 20 m.

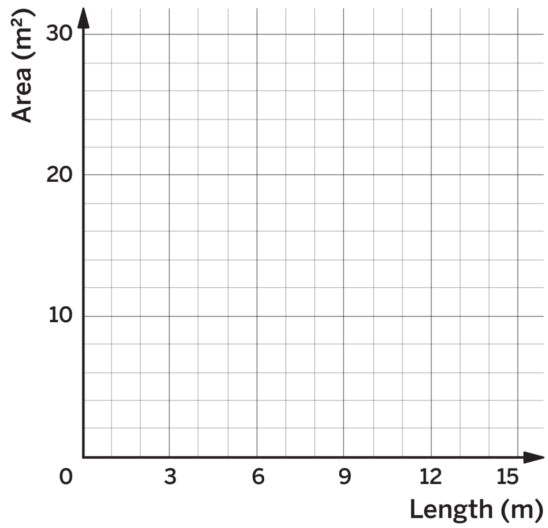
Length (m)	Width (m)	Area (m^2)
1	9	
3	7	
5		
7		
9		

a. Complete the table. What do you notice about the areas?

Name: ______________________ Date: ____________ Period: ____________

b On the coordinate plane, plot the points for the length and area from your table.

c Do the points model a linear relationship? An exponential relationship? Explain your thinking.



3. The table shows the relationship between x and y, the side lengths of a rectangle, and the area of the rectangle. Complete the table.

x (cm)	y (cm)	Area (cm²)
2	4	
4	8	
6	12	
8	16	

Explain why the relationship between x and the area is neither linear nor exponential.

4. Provide a value of r that indicates a line of best fit has a negative slope and models a set of data well.

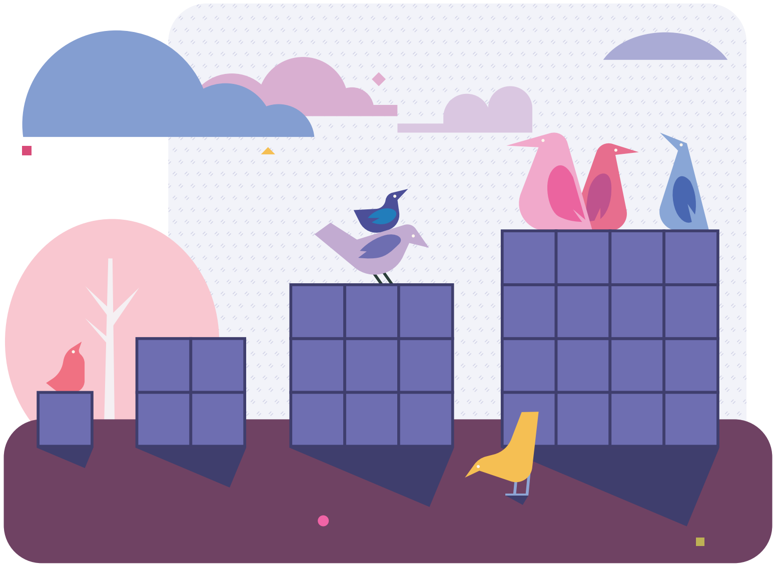
5. Kiran lives 1.5 miles from his school. He walks an average of $\frac{1}{20}$ miles per minute.

a How far is he from his house 10 minutes after leaving school in the afternoon? Show or explain your thinking.

b Write an equation for the distance, in miles, from the school as a function of time t.

How Does It Change?

Let's describe some patterns of change.



Warm-up Notice and Wonder

Study the figures. What do you notice? What do you wonder?

1. I notice . . .

2. I wonder . . .

Log in to Amplify Math to complete this lesson online.

Name: ______ Date: ______ Period: ______

Activity 1 Growing Squares

Study each pattern.

Pattern A

Figure 1 **Figure 2** **Figure 3** **Figure 4**

Pattern B

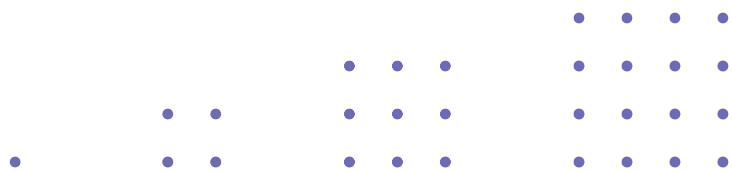


Figure 1 **Figure 2** **Figure 3** **Figure 4**

1. How does each pattern change? Explain your thinking.

2. How would you determine the number of dots in Figure 5 for each pattern? Sketch Figure 5 for each pattern.

3. How would you describe the shape of the figures in Pattern B?

4. In Pattern B, how does the figure number relate to the number of dots in the figure?

Activity 1 Growing Squares (continued)

5. Complete the table with the number of dots used for each figure in the pattern.

Figure number	Number of dots in Pattern A	Number of dots in Pattern B
1		
2		
3		
4		
5		
10		
n		

6. Describe the relationship in Pattern B between the shapes of the figures and the number of dots in Figure n.

A squared variable, by itself or in an expression, is called a ***quadratic*** or ***quadratic expression***. It comes from the Latin *quadrare*, which means "to make square." The expression n^2 is quadratic.

Name: ____________ Date: ________ Period: ________

Activity 2 First and Second Differences

1. Study the figures in each pattern. Do you notice a pattern? Explain your thinking.

Pattern A

Figure 1 **Figure 2** **Figure 3** **Figure 4**

Pattern B

Figure 1 **Figure 2** **Figure 3** **Figure 4**

What do you notice about each pattern? Complete the table to help with your thinking.

Pattern A

Figure number	1	2	3	4
Number of squares				

Pattern B

Figure number	1	2	3	4
Number of squares				

Activity 2 First and Second Differences (continued)

2. Study the relationship between the figure number and number of squares.

a Calculate the difference between the number of squares in each figure.

Pattern A

Figure number	1	2	3	4
Number of squares	1	3	5	7

Difference: $\boxed{}$ $\boxed{}$ $\boxed{}$

Pattern B

Figure number	1	2	3	4
Number of squares	1	4	9	16

Difference: $\boxed{}$ $\boxed{}$ $\boxed{}$

b What do you notice about the difference(s) in the number of squares between Pattern A and Pattern B ?

c For Pattern A, calculate the differences of the differences. That is, subtract the preceding difference in the table from the next difference.

d For Pattern B, calculate the differences of the differences.

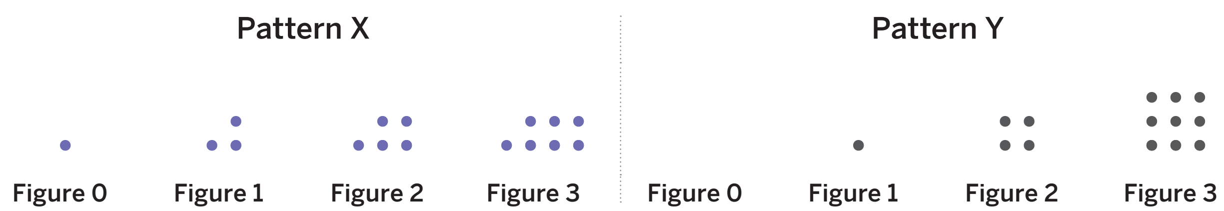
3. What do you notice about the differences of the differences, also known as the *second differences*?

4. Form your own hypothesis about first and second differences for linear and quadratic relationships. Explain your thinking.

Name: .... Date: .... Period: ....

Activity 3 Patterns of Dots

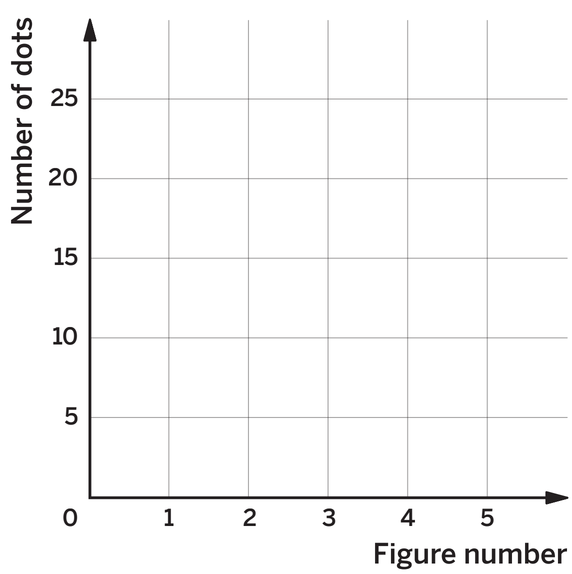
Compare Patterns X and Y.



1. Complete the table with the number of dots in each pattern. Then compare and contrast Patterns X and Y.

Figure	Number of dots in Pattern X	Number of dots in Pattern Y
0		
1		
2		
3		
4		
5		

2. In the graph, plot the number of dots in each figure number in Pattern X and Pattern Y. Use different colors or symbols for each pattern.



3. Does the graph of each pattern confirm your comparison in Problem 1? Explain your thinking.

STOP

Summary

In today's lesson . . .

You observed some patterns that do not change linearly or exponentially. Instead, the change is ***quadratic***, meaning the pattern grows by raising a number or term to the second power, or squaring it. For example, the area of a square with side length n is the ***quadratic expression*** n^2. (The prefix "quad-" means four. While the exponent in quadratic expressions is 2, quadratics are closely related to squares, which have 4 sides.)

You can determine if a pattern is linear or quadratic by analyzing its first and second differences. In a linear relationship, the first differences are equal while the second differences are all 0. In a quadratic relationship, the first differences are not equal, but the second differences are equal.

Reflect:

Name: ________ Date: ________ Period: ________

Practice

1. Pattern A grows by three dots in each successive figure. In Pattern B, the number of dots in each figure is expressed by n^2, where n is the figure number. Sketch Figures 1–3 for each pattern.

Pattern A

Figure 0 Figure 1 Figure 2 Figure 3

Pattern B

Figure 0 Figure 1 Figure 2 Figure 3

2. Examine each pattern.

Pattern Y

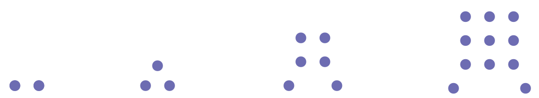


Figure 0 Figure 1 Figure 2 Figure 3

Pattern Z

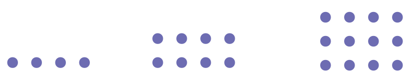


Figure 0 Figure 1 Figure 2 Figure 3

a How many dots will there be in Figure 4 of each pattern?

b Which pattern shows a quadratic relationship between the figure number and the number of dots? Explain your thinking.

Practice

Name: ______________________ Date: __________ Period: __________

3. Select *all* the expressions for the number of dots in a pattern that represent a quadratic relationship with the figure number n.

A. n^2

B. $2n$

C. $n \bullet n$

D. $n + 1$

E. $n + 2$

F. $n \div 2$

4. A garden has a perimeter of 40 ft. Some of the possible measurements are shown in the table.

a Complete the missing measurements in the table.

b What lengths and widths produce the greatest area?

Length (ft)	Width (ft)	Area (ft^2)
4	16	64
8	12	
10		
12		96
14		
		64

5. The function $C(x)$ gives the percentage of homes using only cell phone service x years after 2004. Explain the meaning of each statement.

a $C(10) = 35$

b $C(x) = 10$

c How is $C(10)$ different from $C(x) = 10$?

6. How many small squares will there be in Figure 10?

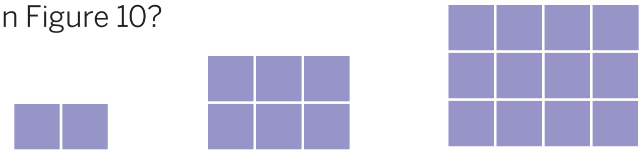


Figure 1 **Figure 2** **Figure 3**

Name: ______ Date: ______ Period: ______

Unit 5 | Lesson 4

Squares

Let's write new quadratic expressions.

Warm-up What Comes Next?

Study the pattern. Sketch or describe how you think Figures 4 and 5 should appear.

Figure 1

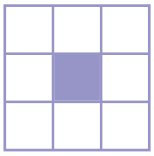


Figure 2

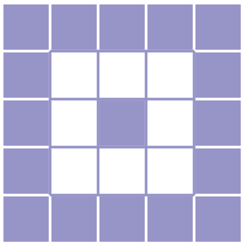


Figure 3

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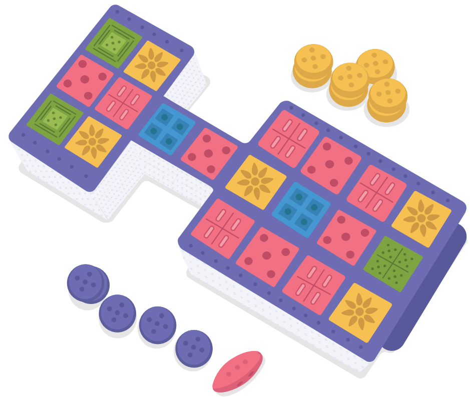
Activity 1 Racing Boards



The Royal Game of Ur, dating back to around 2600 BCE, involved moving game pieces through patterns of 20 squares.

Let's make our own game boards with patterns of squares. Study the pattern.

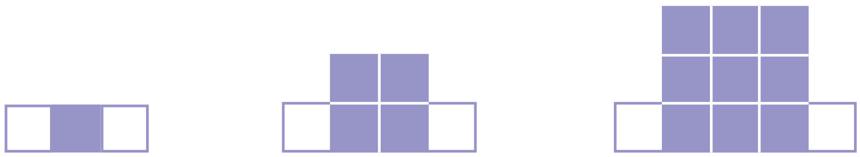


Figure 1 **Figure 2** **Figure 3**

1. Describe the relationship between the figure number n and the total number of squares.

2. Complete the table. In the last row, write a quadratic expression that gives the total number of squares in Figure n.

Figure	Total number of squares
1	
2	
3	
5	
10	
12	
n	

Name: ______________ Date: ______________ Period: ______________

Activity 2 Color Squares

Study the pattern.

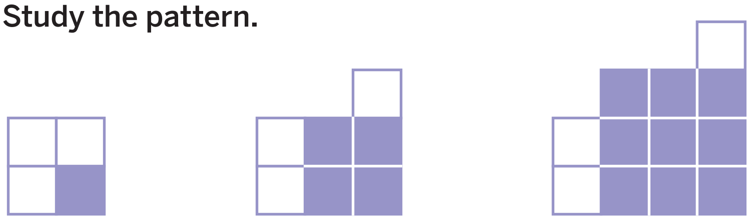


Figure 1 **Figure 2** **Figure 3**

1. How many shaded squares will there be in Figures 4 and 5? How many unshaded squares? Explain or show your thinking.

2. Complete the table for each figure.

	Figure 1	Figure 2	Figure 3	Figure 4	Figure 5
Shaded squares	1				
Unshaded squares	3				

3. Describe the pattern in the number of shaded and unshaded squares in each figure.

4. Write an expression relating the total number of squares in each figure to the figure number n. Use your expression to complete the table. Check your expression by comparing your table values to the actual figures.
Expression:

Figure number	1	2	3	4	5
Total number of squares	4				

Reflect: How did you organize the details within the activity to reach a general conclusion?

Activity 3 Checkerboards

The Royal Game of Ur was likely a direct ancestor of backgammon and checkers. Played in pairs, the player who moved all of their game pieces through the patterns of colored squares first was the winner!

Consider the following figures.

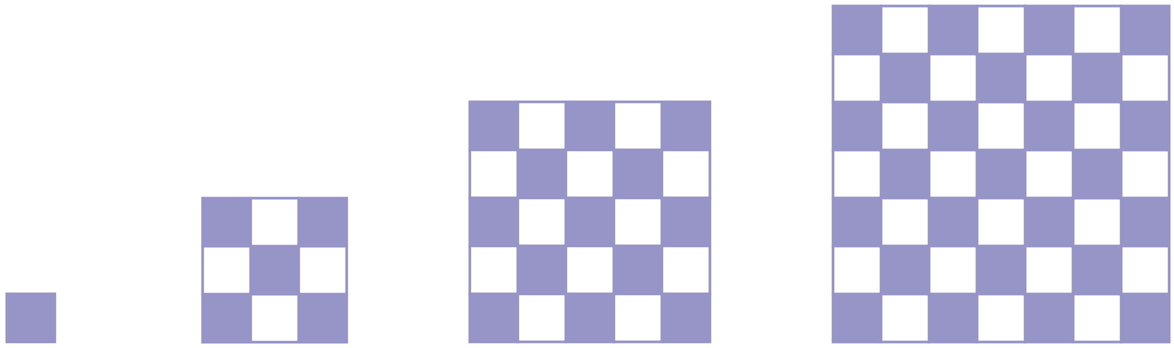


Figure 1 **Figure 2** **Figure 3** **Figure 4**

1. Write a quadratic expression that gives the total number of squares in figure number n.

2. Use this expression to complete the table, noting the calculation you use for each row.

Figure	Calculation	Number of squares
1	$(2 \cdot 1 - 1)^2 = (1)^2$	
2	$(2 \cdot 2 - 1)^2 = (3)^2$	
3		
4		
8		
10		
15		

STOP

Name: .... Date: .... Period: ....

Summary

In today's lesson . . .

You explored more patterns of change that involved squared terms, and you related them to quadratic expressions.

You also explored quadratic relationships in which a constant is added to the squared term. You can write expressions, such as $n^2 + 3$, to represent these relationships or patterns.

Reflect:

Practice

Name: ______________________ Date: ____________ Period: ________

1. Study the pattern of dots.

Figure 0 Figure 1 Figure 2 Figure 3

a Complete the table.

Figure	Total number of dots
0	
1	
2	
3	

b How many dots will there be in Figure 10?

c How many dots will there be in Figure n?

2. Study the sequence of figures. Write a quadratic expression to describe the number of squares in Figure n. Explain your thinking.

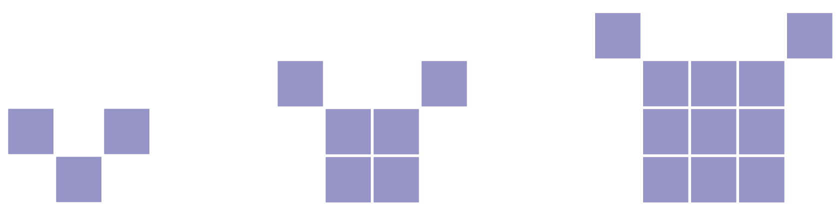


Figure 1 Figure 2 Figure 3

Name: ________________ Date: ________ Period: ________

Practice

3. Consider a pattern that starts with the terms $1, \frac{5}{2}, \ldots$ Determine the next three numbers of the pattern if the pattern grows . . .

a Linearly

b Exponentially

4. Here are some lengths and widths of a rectangle whose perimeter is 20 m.

a Complete the table. What do you notice about each area?

Length (m)	Width (m)	Area (m^2)
1	9	
3	7	
5		
7		
9		
ℓ		

b Predict whether the area of the rectangle will be greater or less than 25 m^2 if the length is 5.25 m.

c Write a quadratic expression for the rectangle with a perimeter of 40 m, whose dimensions yield the greatest area.

Seeing Squares as Functions

Let's describe some other geometric patterns.

Warm-up Area and Perimeter of Squares

Figures A, B, and C are shown. Figure A is a square and Figures B and C are copies of Figure A with a unit square either added or removed.

Figure A

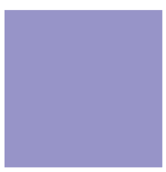
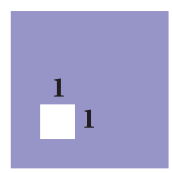


Figure B

Figure C



1. Calculate the perimeter and area of each figure if the side length of the larger square is 5 units:

   Perimeter of Figure A = 　　　　Area of Figure A =

   Perimeter of Figure B = 　　　　Area of Figure B =

   Perimeter of Figure C = 　　　　Area of Figure C =

2. Write an expression for the perimeter and area of each figure if the side length of the larger square is x units:

   Perimeter of Figure A = 　　　　Area of Figure A =

   Perimeter of Figure B = 　　　　Area of Figure B =

   Perimeter of Figure C = 　　　　Area of Figure C =

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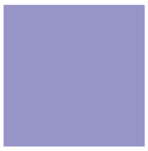
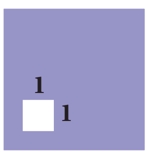
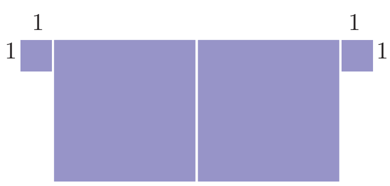
Name: ______________________ Date: __________ Period: __________

Activity 1 Squares on Squares

1. Write an expression to represent the area (in square units) of the shaded parts of each figure, given the side lengths, of the larger square.

Figure	Side length of 4	Side length of $4x$	Side length of $(x + 3)$
			
			
			
			
			

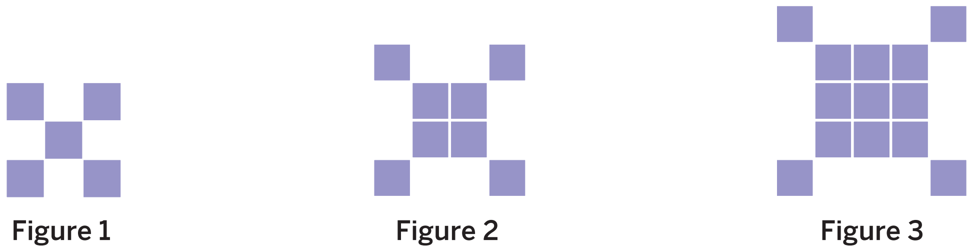
2. The side length of a large square is x units. Sketch a figure the total area represented by each expression.

a $x^2 + 2$

b $x^2 - 2$

Activity 2 Expanding Squares

Figures 1, 2, and 3 are growing. Assume the pattern continues.



1. What would Figures 4 and 5 look like?

   a. Sketch or describe Figures 4 and 5.

   b. How many unit squares are in each of these figures? Explain or show your thinking.

2. What would Figures 10 and 18 look like?

   a. Sketch or describe Figures 10 and 18.

   b. How many unit squares are in each of these figures? Explain or show your thinking.

Name: ______ Date: ______ Period: ______

Activity 2 Expanding Squares (continued)

3. Write a function to represent the relationship between the figure number n and the number of unit squares $S(n)$.

4. Explain how each part of your function in Problem 3 relates to the given visual pattern.

5. The function $V(n) = n^2 - 1$ represents a pattern of unit squares. Sketch the first three figures of the pattern.

Are you ready for more?

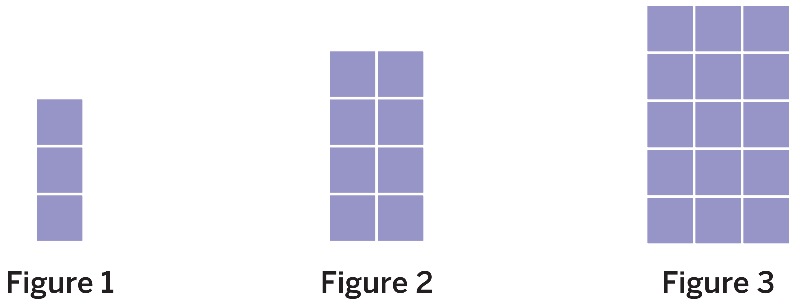
1. For the original pattern of figures, write a function to represent the relationship between the figure number n and the perimeter $P(n)$.

2. For the pattern you created in Problem 5 of the activity, write a function to represent the relationship between the figure number n and the perimeter $P(n)$.

3. Are these linear functions?

Activity 3 Expressing Quadratic Patterns

Study the figures. Assume the pattern continues.



1. Kiran says that the pattern grows linearly. He reasons that as the figure number increases by one, the number of rows and the number of columns also increase by one. Do you agree? Explain your thinking.

2. To represent the number of squares in the figure number n, Diego and Jada wrote different functions.

- Diego wrote the function $f(n) = n(n + 2)$.
- Jada wrote the function $g(n) = n^2 + 2n$.

Who is correct? Diego or Jada? Explain your thinking.

STOP

Name: ______________________ Date: __________ Period: __________

Summary

In today's lesson . . .

You discovered that sometimes a quadratic relationship is expressed without a squared term.

Suppose the length of a rectangle is n and its width is $n + 1$. The rectangle's area is then $n \cdot (n + 1)$, which is equivalent to $n^2 + n$ by the Distributive Property. Both of these expressions for area are quadratic expressions.

In the examples you saw, the relationship between the figure number and the number of squares can be modeled by the ***quadratic function*** $f(n)$ whose input value n is the figure number, and whose output value is the number of squares in Figure n. This function can be defined by $f(n) = n(n + 1)$ or $f(n) = n^2 + n$. A quadratic function can be represented with an equation, a table of values, a graph, or a description.

Reflect:

Practice

Name: ________ Date: ________ Period: ________

1. Study the figures. Assume the pattern continues.

Figure 1

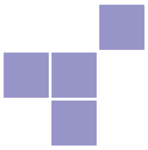


Figure 2

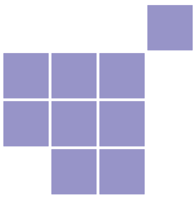


Figure 3

a Sketch or describe Figures 4 and 15.

b How many unit squares will there be in each of these figures?

2. Write a function to represent the relationship between the figure number n and the number of unit squares $f(n)$ in each figure in Problem 1.

a $f(n) =$ ________

b Explain how your function relates to the pattern.

3. Each figure shown is composed of large and small squares. The side length of each large square is x.

Figure A

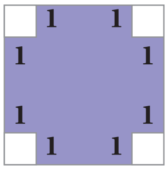
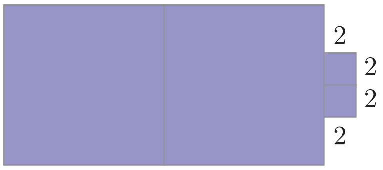


Figure B



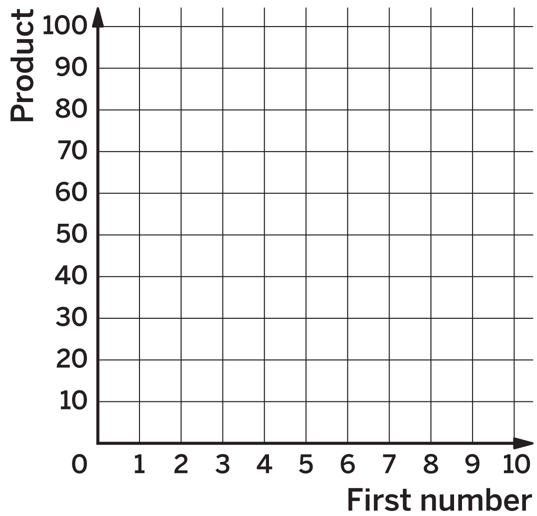
Figures are not drawn to scale.

Write an expression for the area of the shaded part of each figure.

Name: ______________________ Date: __________ Period: __________

4. For each row, determine the product of each pair of numbers. Then plot the points to show the relationship between the first number and the product.

First number	Second number	Product
1	6	
2	7	
3	8	
5	10	
7	12	



5. Is the relationship between the first number and the product in Problem 4 exponential? Explain your thinking.

6. Which expression represents the total number of squares in Figure n?

Figure 1 Figure 2 Figure 3

A. $n^2 + 1$

B. $n^2 - 1$

C. $n^2 - n$

D. $n^2 + n$

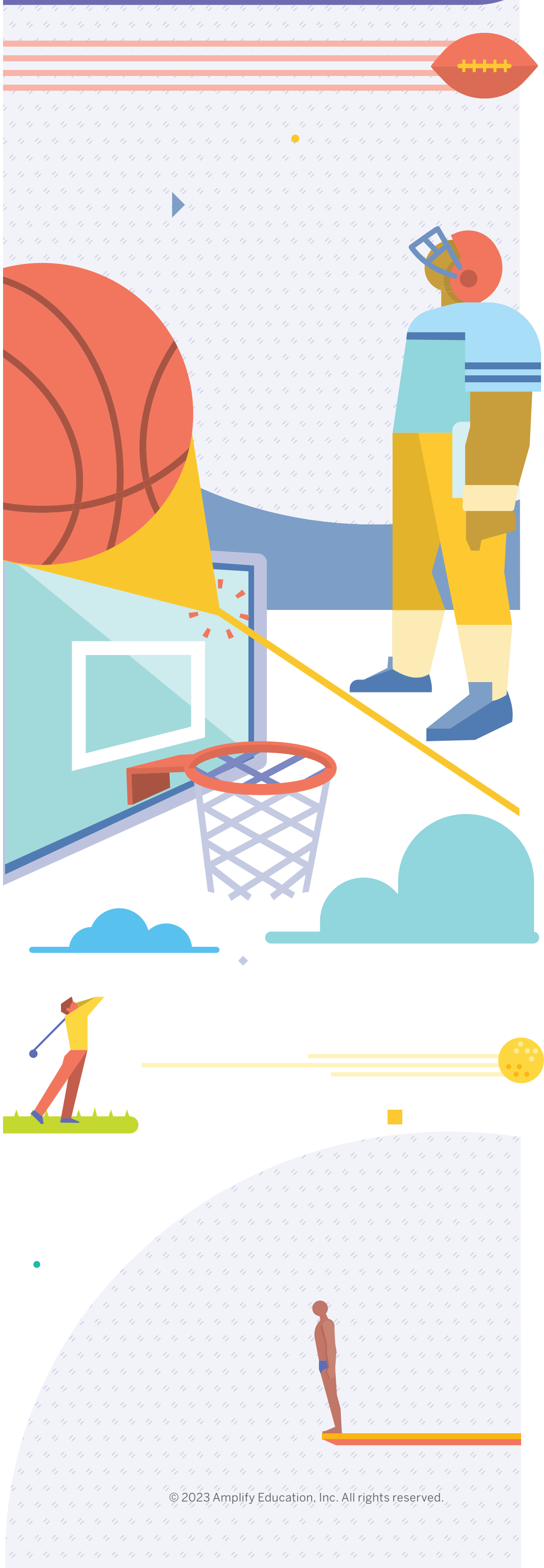
My Notes:

SUB-UNIT

2 Quadratic Functions

Narrative Connections

What would sports be like without quadratics?



In a word: *dull.*

Imagine a quarterback throwing a pass, only to have it sail serenely over the receiver's head in a straight line. Or a basketball shot from half-court that hopelessly deflects off the backboard.

Shot puts would not land. Olympic divers would be awkwardly stranded on their diving boards. Racers would always be on cruise control.

Imagine all the nice arcs we are used to seeing replaced by hard, straight lines. Sure, it might be funny the first few times, but pretty soon the fans would be asleep in their seats.

Quadratics bring spice to every sport.

It is what allows balls to land, and bodies to accelerate during freefall. Without quadratics, there is no excitement, no mystery. Will a free throw swish or miss? Can a batter clear the fence for a homerun?

Quadratics (and an insane amount of skill) let record-shattering gymnast Simone Biles land her triple-twisting double backflip. They're what allowed legendary shot putter Randy Barnes to hurl a 16-pound ball more than 23 meters.

All the excitement and tension we feel in these great moments of sports history are thanks in part to quadratics. While most of us probably do not watch sports with pencil and paper in our laps, we still experience these quadratic relationships, from something as simple as a game of catch to the breaking of a world record.

Unit 5 | Lesson 6

Comparing Functions

Let's compare quadratic and exponential changes and see which grows faster.

Warm-up From Least to Greatest

The table shows a variety of sports statistics from 2018 and 2019.

	Expression	Statistical description
A	$1.17 \cdot 10^5$	Number of high school female athletes in the United States in 2018
B	$1.448 \cdot 10^{10}$	Total revenue of all National Football League teams in 2018
C	$3.8 \cdot 10^5$	Average salary of a Major League Soccer player in 2018
D	$4.141 \cdot 10^8$	Estimated number of people who watched the 2019 Women's World Cup tournament
E	$9 \cdot 10^4$	Average salary of a Canadian Football League player in 2018

1. List each row in order by value, from least to greatest, without evaluating the expressions.

2. What strategy did you use to compare and order the rows, without evaluating each expression?

Log in to Amplify Math to complete this lesson online.

Name: ______________________ Date: ____________ Period: __________

Activity 1 Which One Grows Faster?

In Pattern A, the lengths and widths of the figures each grow by one unit square from one figure to the next. In Pattern B, the number of unit squares doubles from each figure to the next. In each pattern, the number of unit squares is a function of the figure number n.

Co-craft Questions:
Before you begin this activity, study these patterns. Work with your partner to write 2–3 mathematical questions you have about this scenario.

Pattern A

Figure 1 **Figure 2** **Figure 3**

Pattern B

Figure 1 **Figure 2** **Figure 3**

1. Sketch Figures 4 and 5 of Pattern A.

a. Write a function $f(n)$ to represent the number of unit squares in Figure n of Pattern A.

b. Is the function linear, quadratic, or exponential?

2. Sketch Figures 4 and 5 of Pattern B.

a. Write a function $g(n)$ to represent the number of unit squares in Figure n of Pattern B.

b. Is the function linear, quadratic, or exponential?

Activity 1 Which One Grows Faster? (continued)

3. Complete the table for each pattern.

Pattern A

Figure number, n	Number of squares, $f(n)$
1	
2	
3	
4	
5	
6	
7	
8	

Pattern B

Figure number, n	Number of squares, $g(n)$
1	
2	
3	
4	
5	
6	
7	
8	

4. How would the two patterns compare if they continue to grow? What observations can you make?

Name: ______ Date: ______ Period: ______

Activity 2 Comparing Two Functions

When writing computer code that performs a task or procedure with a data set, computer scientists often study how long it takes to run the code as a function of the data set's size. If this function grows slowly, that means the algorithm works quickly, even for large data sets. But sometimes scientists, such as Tibor Radó, come across functions like the "busy beaver function," which grows very quickly.

For now, consider two functions: $p(x) = 6x^2$ and $q(x) = 3^x$.

Investigate the output of $p(x)$ and $q(x)$ for different values of x. Which function will have a greater value as x increases? Support your response with tables, graphs, or other representations.

Featured Mathematician

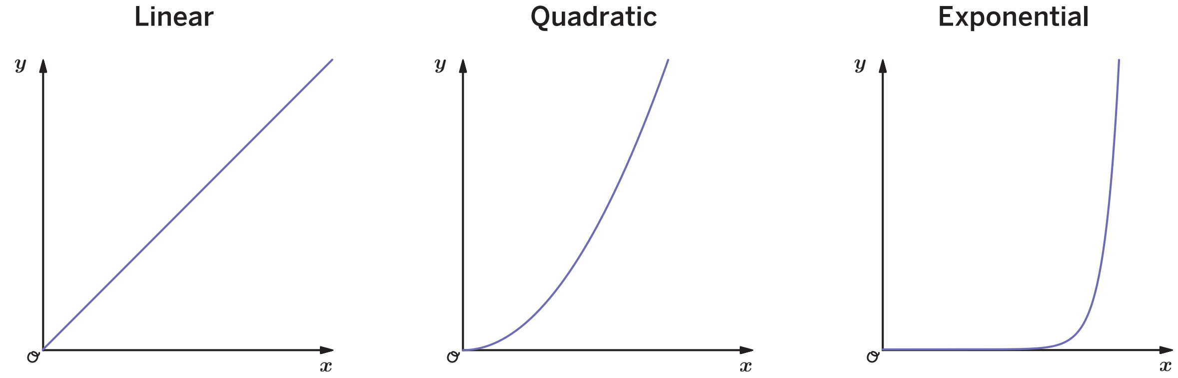
Tibor Radó

Tibor Radó was born in Hungary in 1895 and moved to the U.S. in 1929. He taught for many years at Ohio State University and served as a science consultant to the U.S. government during World War II. In 1962, he wrote a paper in which he introduced the world to the "busy beaver function." Not only did he prove that this function grew faster than quadratics and exponentials — he proved it grew faster than any "computable function," which includes any function you have seen up to this point.

The busy beaver function is usually written as $\Sigma(x)$. It starts out mildly enough: $\Sigma(1) = 1$, $\Sigma(2) = 4$, $\Sigma(3) = 6$, and $\Sigma(4) = 13$. But no one knows the precise value of $\Sigma(5)$. What about $\Sigma(6)$? It is greater than 3.5×10^{18267}. And $\Sigma(7)$? It. Is. So. Big.

Activity 3 Functions Have Sound

The graphs of three functions are shown.



1. If each type of function had a sound, what do you imagine it would sound like? Explain your thinking.

   a. Linear functions would sound like . . .

   b. Quadratic functions would sound like . . .

   c. Exponential functions would sound like . . .

2. Your teacher will play five sounds. Match each sound with one equation and one function type listed in the table.

Equations	Function types
$y = -x^2 + 10$	Linear
$y = x^2$	Quadratic
$y = 2^x$	Exponential
$y = 2x$	

   a. Sound 1: Equation ________ Function type ________

   b. Sound 2: Equation ________ Function type ________

   c. Sound 3: Equation ________ Function type ________

   d. Sound 4: Equation ________ Function type ________

STOP

Name: Date: Period:

Summary

In today's lesson . . .

You compared increasing quadratic and exponential functions.

Exponential functions, like $g(x) = 3^x$, always increase by the same factor (in this case, 3). But quadratic functions, like $f(x) = 6x^2$, increase by different factors depending on the value of x.

Sooner or later, exponential growth always overtakes quadratic growth.

Reflect:

Practice

Name: ______ Date: ______ Period: ______

1. Consider the exponential function $f(x) = 1.5^x$ and the quadratic function $g(x) = 500x^2 + 345x$. As the values of x increase, which function will eventually have a greater value? Explain your thinking.

2. Make a table of values to show that the values of the exponential expression $3(2)^x$ eventually overtake the values of the quadratic expression $3x^2 + 2x$.

3. The table shows values of the expressions $10x^2$ and 2^x for different values of x.

x	$10x^2$	2^x
1	10	2
2	40	4
3	90	8
4	160	16

a. Describe how the values of each expression change as x increases.

b. Predict which expression will have a greater value when x is 8, 10, and 12.

c. Determine the value of each expression when x is 8, 10, and 12.

d. Make a conjecture about how the values of the two expressions change as x becomes greater and greater.

Name: .... Date: .... Period: ....

Practice

4. Refer to the pattern of shapes. The area of each small square is 1 cm^2.

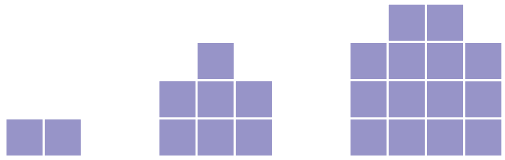


Figure 1 Figure 2 Figure 3

a What is the area of Figure 10?

b What is the area of Figure n?

c Describe how the pattern is growing.

5. The height, in meters, of a bungee jumper during freefall is given by the function $h(t) = -4.9t^2 + 83$, where t is time in seconds. Select *all* statements that are true.

A. The initial height of the bungee jumper is 83 meters.

B. The height of the bungee jumper after 1 second is 78.1 meters.

C. The bungee jumper will take 78.1 seconds to reach a height of 0 meters.

D. $h(0) = 83$ means the bungee jumper will have a height of 0 meters after 83 seconds.

Unit 5 | Lesson 7

Building Quadratic Functions to Describe Falling Objects

Let's measure falling objects.

Warm-up Notice and Wonder

Study the table. What do you notice? What do you wonder?

x	0	1	2	3	4	5
y	0	16	64	144	256	400

1. I notice . . .

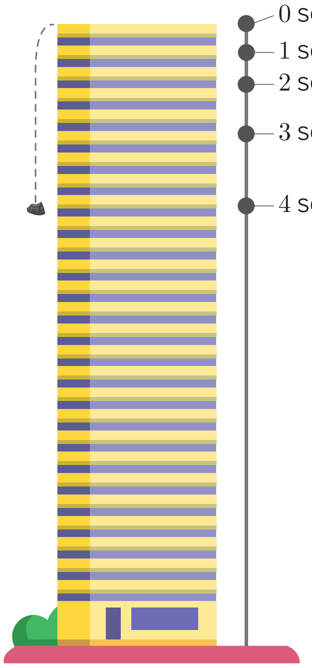
2. I wonder . . .

Log in to Amplify Math to complete this lesson online.

Name: ______ Date: ______ Period: ______

Activity 1 Falling From the Sky

A rock is dropped from the top floor of a 1,000-foot-tall building. A camera captures the distance the rock traveled, in feet, after each second.



1. Jada noticed that the distances fallen are all multiples of 16. She wrote down:

   $16 = 16 \cdot 1$

   $64 = 16 \cdot 4$

   $144 = 16 \cdot 9$

   $256 = 16 \cdot 16$

   She also noticed that 1, 4, 9, and 16 can be written as 1^2, 2^2, 3^2, and 4^2.

   Use Jada's observations to predict the distance fallen after 5 seconds.

2. How far will the rock have fallen after 10 seconds? How far from the ground will it be? Explain or show your thinking.

3. Write a function $d(t)$ that represents the distance fallen after t seconds.

Activity 2 Egg Drop

Let's investigate free-falling objects a bit further. In this experiment, you will investigate the relationship between speed and the distance an object falls. Your goal is to determine the maximum height from which a hard-boiled egg can be dropped without breaking. Your group will be given two eggs, a measuring tape, a 6.5 ft piece of aluminum foil, and a stopwatch.

1. Lay your piece of aluminum foil on the floor so that one edge is up against the wall. Use the measuring tape and mark a "starting point" that is 6 ft from the wall with tape or a pen.

2. Determine the greatest speed at which you can roll the egg from the starting point, so the egg does not break. You *only* have two eggs to use for your trials, so be careful. (*Tip: Try rolling the egg slowly at first before trying faster speeds*.)

a For each roll, you and your partner will take turns rolling the egg and recording how many seconds it takes for the egg to hit the wall. Complete the first two columns of the table.

b Calculate the speed of each roll by dividing the distance the egg travels (6 ft) by the time (in seconds) it takes. Record this speed in the third column of the table.

Roll number	Time for egg to hit the wall (seconds)	Did it break?	Speed of the egg (ft/second)
1			
2			
3			
4			
5			
6			

Name: .... Date: .... Period: ....

Activity 2 Egg Drop (continued)

3. Refer to the table. What was the fastest speed at which the egg did *not* break?

4. Let v represent your speed from Problem 3. Now let's try dropping the egg onto the floor.

a The distance a falling object travels before reaching a speed v is given by the expression $\frac{v^2}{64}$. Use your response from Problem 3 to determine the greatest height from which you can drop your egg so that it will not break.

b Use the measuring tape to measure this distance above the floor. Be sure of the location, steady your hand, then drop your egg. Did it break?

Activity 3 Galileo and Gravity

Galileo Galilei, an Italian scientist of the 1600s, was not afraid to challenge the popular scientific beliefs of his time. For his efforts, he was imprisoned, his book was banned, and he spent his final days under house arrest.

Galileo and other scientists also studied the motion of free-falling objects. The law they discovered is represented by the equation $d = 16 \cdot t^2$, which gives the distance fallen d, in feet, as a function of time t, in seconds.

1. An object is dropped from a height of 576 ft. How far does it fall in 0.5 seconds?

Galileo concluded that objects of any size fall at the same rate. (Prior philosophers, like Aristotle, thought heavier objects fell faster.) To see if Galileo was right, Jada drops a heavy rock and a light rock from the same height.

2. Complete the tables representing Jada's observations of each rock, using the equation $d = 16 \cdot t^2$.

Heavy Rock

Time (seconds)	Distance traveled (ft)
0	0
1	16
2	64
3	
4	
t	

Light Rock

Time (seconds)	Distance from the ground (ft)
0	576
1	560
2	512
3	
4	
t	

Name: .... Date: .... Period: ....

Activity 3 Galileo and Gravity (continued)

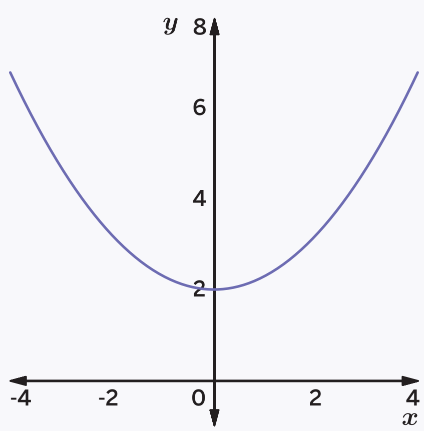
3. How are the two tables similar? How are they different?

4. Refer to the table for the light rock. What does each term of the expression in the last row represent?

Are you ready for more?

Galileo correctly observed that gravity causes objects to fall in such a way so that the distance fallen is a quadratic function of the time elapsed. He got a little carried away, however, and assumed that a hanging rope or chain could also be modeled by a quadratic function.

Here is a graph of such a shape (called a *catenary*) along with a table of approximate values.



x	-3	-2	-1	0	1	2	3
y	4.70	3.09	2.26	2	2.26	3.09	4.70

Mai thought that the graph of the catenary could be modeled by a quadratic function of the form $y = ax^2 + 2$, where a is constant. Is Mai correct? Explain your thinking.

STOP

Summary

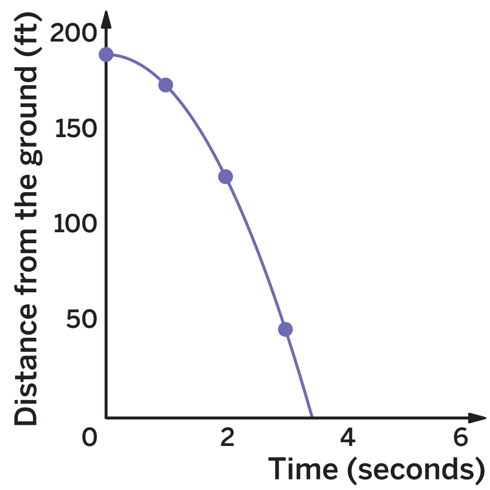
In today's lesson . . .

You saw that the distance traveled by a falling object can be represented by a quadratic function of time.

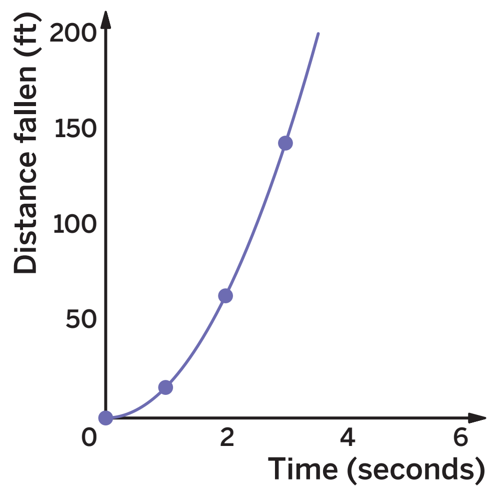
The table shows the distance d an object has fallen after t seconds, as well as the object's distance from the ground, or height h. This object was dropped from an initial height of 190 ft.

Time, t (seconds)	Distance fallen, d (ft)	Height, h (ft)
0	0	190
1	16	174
2	64	126
3	144	46
t	$16t^2$	$190 - 16t^2$

Here is the graph of the relationship between t and h.



Here is a graph of the relationship between t and d.



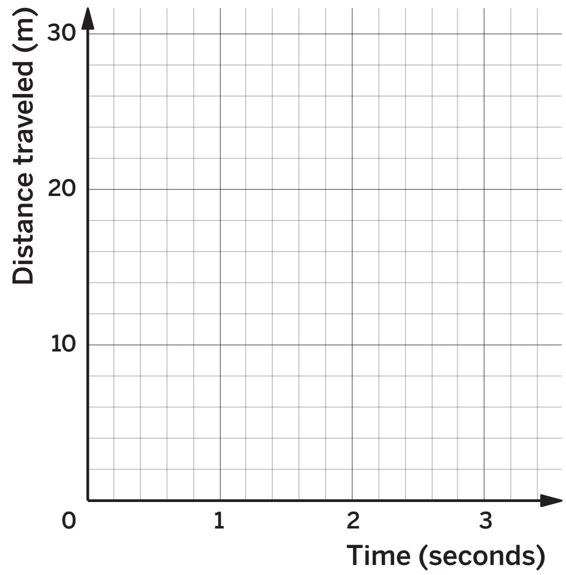
Reflect:

Name: .... Date: .... Period: ....

Practice

1. A baseball has traveled d meters, t seconds after being dropped from the top of a building. The distance traveled by the baseball can be modeled by the equation $d = 5 \bullet t^2$. Complete the table and plot the data on the coordinate plane.

t (seconds)	d (m)
0	
0.5	
1	
1.5	
2	



2. A rock is dropped from a bridge over a river. Which column of data could represent the distance fallen, in feet, as a function of time, in seconds?

Time (seconds)	Distance A fallen (ft)	Distance B fallen (ft)	Distance C fallen (ft)	Distance D fallen (ft)
1	0	0	180	180
2	48	16	132	164
3	96	64	84	116
4	144	144	36	36

3. Determine which function, $f(n) = 5n^2$ or $g(n) = 3^n$, will have the greater value when:

a. $n = 1$

b. $n = 3$

c. $n = 5$

Name: .... Date: .... Period: ....

4. Select *all* the expressions that give the number of small squares in Figure n.

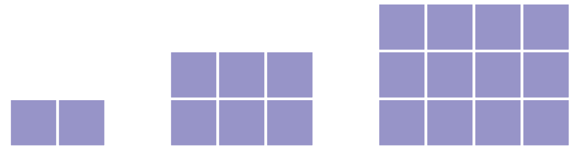


Figure 1 Figure 2 Figure 3

A. $2n$

B. n^2

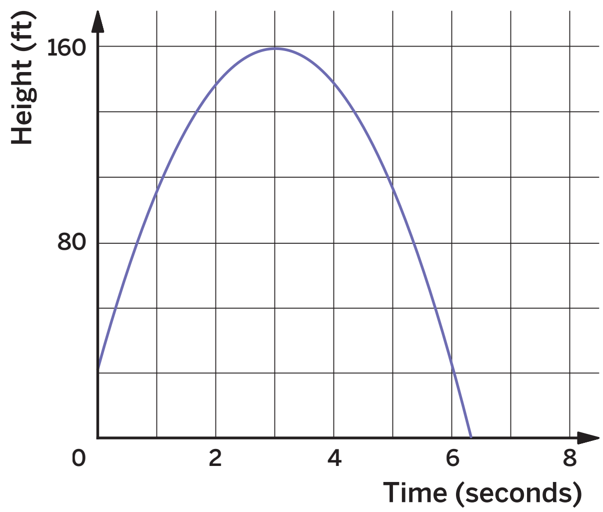
C. $n+1$

D. n^2+1

E. n^2+n

F. $n(n+1)$

5. A rocket is launched in the air and its height, in feet, is modeled by h as a function of time, in seconds. Here is a graph representing h. Select *all* true statements about the scenario.



A. The maximum height of the rocket is about 160 ft.

B. The rocket is launched from about 50 ft above the ground.

C. The rocket reaches its maximum height at about 3 seconds.

D. The rocket reaches its maximum height at about 160 seconds.

E. The rocket is launched from a height less than 40 ft above the ground.

Name: ______________ Date: ______ Period: ______

Unit 5 | Lesson 8

Building Quadratic Functions to Describe Projectile Motion

Let's study objects being launched in the air.

Warm-up Notice and Wonder

Read the image descriptions. What do you notice? What do you wonder?

Edward H. White II, the first American astronaut to perform a spacewalk (1962).

PaoloBona/Shutterstock.com

Bruce McCandles II, the first astronaut ever to perform an untethered spacewalk (1984).

PaoloBona/Shutterstock.com

1. I notice . . .

2. I wonder . . .

Log in to Amplify Math to complete this lesson online.

Plan ahead: What emotions are stirred when you look at the activity? How will you manage them?

Activity 1 Tracking Foofoo's Flight

At the beginning of this unit, you predicted the trajectory of Foofoo the clown after being fired out of a cannon, both on Earth and on the Moon. Now that you know more about quadratic functions, let's revisit Foofoo.

Foofoo is loaded into a cannon that is 10 ft above the ground. He is launched straight up at a speed of 406 ft per second. Imagine that there is no gravity and that Foofoo travels upward at a constant speed.

1. Complete the table with the heights that Foofoo reaches at different times.

Time (seconds)	0	1	2	3	4	5
Height (ft)	10					

2. Write a function to model Foofoo's height $n(t)$, in feet, t seconds after he is launched from the cannon. Again, assume there is no gravity.

3. What type of function can be used to model Foofoo's flight? What does each term of your function represent in context?

In reality, Foofoo is launched from a cannon on Earth, where gravity cannot be ignored. The table shows the heights that Foofoo reaches at different times after being launched from the cannon.

Time (seconds)	0	1	2	3	4	5
Height (ft)	10	400	758	1,084	1,378	1,640

4. Compare the values in each table. What do you notice?

Name: ________________ Date: ________ Period: ________

Activity 1 Tracking Foofoo's Flight (continued)

5. Complete the table to determine the differences in Foofoo's heights after each launch for $t = 0, 1, 2, 3$, and 4.

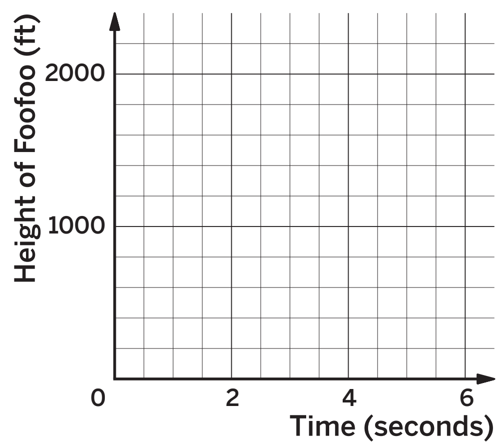
t	Height without gravity	Height with gravity	Difference in height
0	10	10	0
1			
2			
3			
4			

6. Study the last column. Where have you seen this pattern before? Write an expression that represents the difference in heights after t seconds.

7. Plot each set of data from Foofoo's flights on the coordinate plane. Use x's to represent the data without gravity and dots to represent the data with gravity.

a How are the graphs similar? How are they different?

b Use the graph to describe the effect of Earth's gravity on Foofoo's height over time.



8. Your function $n(t)$ from Problem 2 does not account for the effect of gravity.

a Write a function that models the height of Foofoo $d(t)$, in feet, t seconds after being launched from the cannon on Earth (i.e., with gravity).

b In your own words, what is the meaning of each term in your function?

Activity 2 Tracking a Cannonball

Foofoo decides it is safer to launch cannonballs rather than himself. The function $g(t) = 50 + 312t - 16t^2$ gives the height, in feet, of a cannonball t seconds after the ball leaves the cannon.

1. What information do you think each term of $g(t)$ provides about the cannonball?

2. Use graphing technology to graph $g(t)$. Adjust the axes limits to include these boundaries: $0 < x < 25$ and $0 < y < 2000$.

   a. Describe the shape of the graph. What information does the shape provide about the movement of the cannonball?

   b. Approximate the greatest height the cannonball reaches.

   c. Estimate the time the cannonball reaches its greatest height.

   d. Estimate the time the cannonball hits the ground.

3. What is an appropriate domain for this function? Explain your thinking.

Are you ready for more?

If the cannonball was launched at 800 ft/second, would it reach a mile in height? Explain your thinking. (1 mile $= 5{,}280$ ft)

STOP

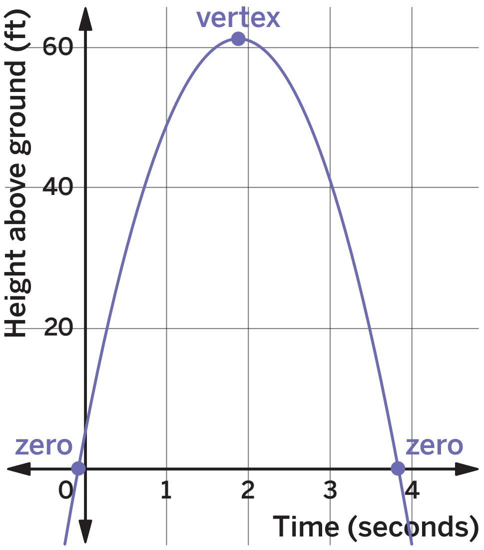
Name: ______ Date: ______ Period: ______

Summary

In today's lesson . . .

You looked at the height of objects that are launched upward and then fall because of gravity.

If $h(t)$ is the height of the object at time t, this function will have the form $h(t) = c + bt - 16t^2$. The value c is the vertical intercept and represents the initial height of the object, while the value b represents the initial vertical speed at which the object is launched. The term $-16t^2$ represents the effect of gravity, which pulls the object down.



The graph shows a typical trajectory. The object is launched from an initial height of 5 ft at $t = 0$ until it reaches its maximum height, represented by the ***vertex***. The object then falls down until it reaches the ground, represented by the horizontal intercept at approximately 3.8 seconds. This intercept is also called a ***zero*** of the function, because it is where $h(t)$ equals 0. This span of time between 0 and 3.8 seconds is an appropriate domain for the function because any values outside of this would have no meaning in context.

Reflect:

Practice

Name: ______ Date: ______ Period: ______

1. The height, in meters, of a diver above the water is given by the function $h(t) = -5t^2 + 10t + 3$, where time t is time measured in seconds. Select *all* statements that are true.

   A. The graph that represents $h(t)$ starts at the origin and curves upward.

   B. The diver begins at the same height as the water level.

   C. The function has one zero that makes sense in this situation.

   D. The function has two zeros that make sense in this situation.

   E. The diver begins 3 m above the water.

   F. The diver begins 5 m above the water.

2. *Technology required*. Two rocks are launched straight up in the air. The height of Rock A is given by the function $f(x) = 4 + 30t - 16t^2$. The height of Rock B is given by the function $g(t) = 5 + 20t - 16t^2$. In both functions, t represents the time in seconds, and the height is measured in feet. Use graphing technology to graph each function. Determine which rock hits the ground first and explain your thinking.

3. Each function represents an object's distance from the ground in meters as a function of time t, in seconds.

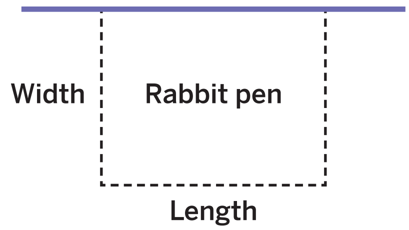
   Object A: $j(t) = -5t^2 + 25t + 50$ **Object B:** $k(t) = -5t^2 + 50t + 25$

   a Which object was launched with the greater vertical speed?

   b Which object was launched from the greater height?

Name: .... Date: .... Period: ....

4. Tyler is building a pen for his rabbit on the side of the garage. He needs to fence three sides of his lawn and wants to use 24 ft of fencing.



a The table shows some possible lengths and widths of the fence. Complete the table by calculating the possible areas.

Length (ft)	Width (ft)	Area (ft^2)
8	8	
10	7	
12	6	
14	5	
16	4	

b Which dimensions should Tyler choose to give his rabbit the most room?

5. Refer to the pattern of dots.

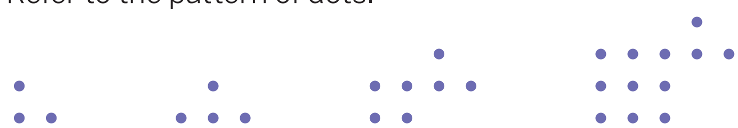


Figure 0 Figure 1 Figure 2 Figure 3

Figure	Number of dots
0	
1	
2	
3	

a Complete the table.

b How many dots will there be in Figure 10?

c How many dots will there be in Figure n?

6. Han bought a bus pass for \$15. Each bus ride costs \$1.50. The expression $15 - 1.50r$ represents the dollar amount left on Han's bus pass after r rides.

a What does 15 represent in this context?

b What does $1.50r$ represent in this context?

c Why is the expression $15 - 1.50r$ and not $15 + 1.50r$?

d If the expression $15 - 1.50r$ is factored to become $1.50(10 - r)$, what does $(10 - r)$ represent?

Unit 5 | Lesson 9

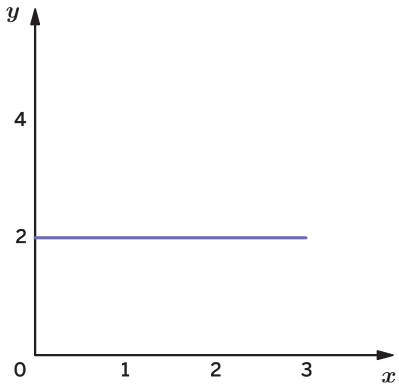
Building Quadratic Functions to Maximize Revenue

Let's study how to maximize revenue.

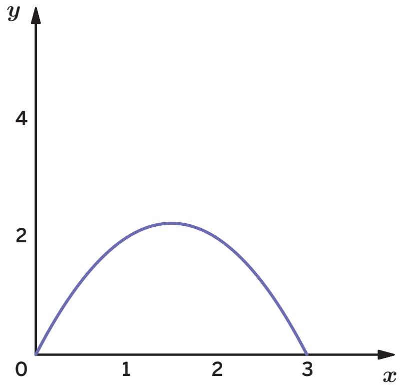
Warm-up Which One Doesn't Belong?

Which of the following graphs does not belong? Use your knowledge of the key features of quadratic functions and graphs in general to explain your thinking.

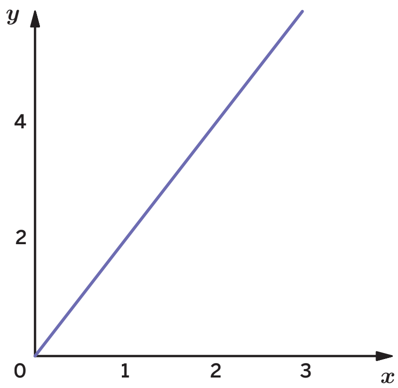
A.



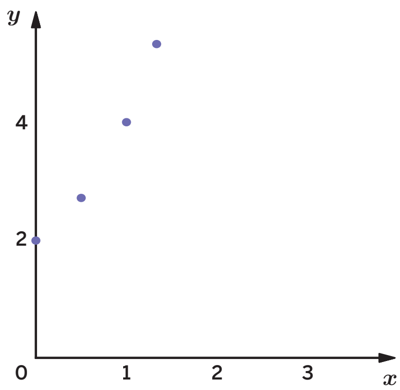
C.



B.



D.



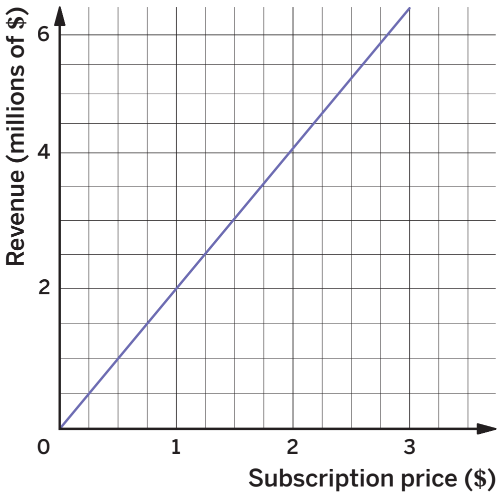
Log in to Amplify Math to complete this lesson online.

Name: ______________________ Date: __________ Period: __________

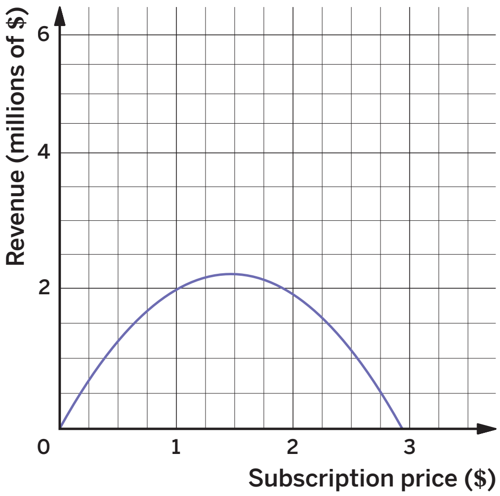
Activity 1 The Rise of Streaming

Kiran works for a streaming service company, DashTV. His team just released an option where customers can put on special glasses to watch all their programming in 3D. His team created two models to estimate the revenue of this new product — a linear model and a quadratic model — which show the daily revenue generated for different subscription prices.

Model A predicts that for every $1 increase of the subscription price, the revenue will increase $2 million per day.



Model B predicts that revenue will initially increase, but after peaking at a maximum value the revenue decreases.



1. Which of Kiran's models do you think is more realistic? Explain your thinking.

2. Model B represents the function $f(x) = 3x - x^2$ where $f(x)$ is the amount of revenue generated (in millions of dollars) per day when the subscription price is x dollars.

a What are some real-world events that could cause this graph to curve downward?

b Estimate the vertex of this graph. What does it represent in this context?

c What does the domain $0 \leq x \leq 3$, of the graph represent?

Activity 2 What Price to Charge?

Kiran starts his own company, which allows fans to watch Country Football League (CFL) games online. Kiran must decide how much customers should pay to watch a single CFL game.

Based on competitors' data, Kiran predicts that if he charges x dollars per game, then the average number of CFL games bought, in thousands, is $18 - x$.

1. Complete the table to show the number of predicted CFL games purchased at each price and its corresponding predicted revenue.

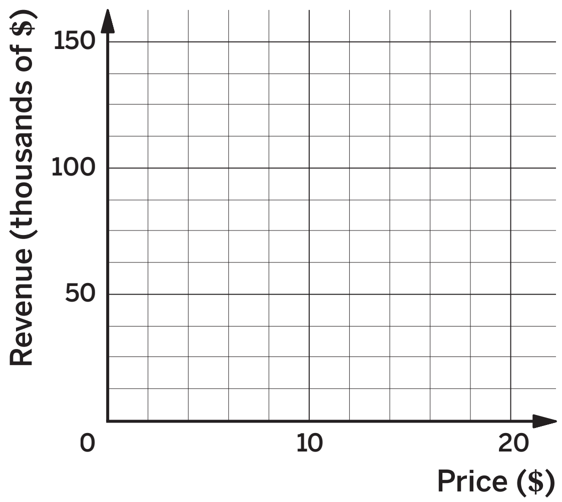
Price ($)	Number of games purchased (thousands)	Revenue (thousands of $)
3	15	45
5		
10		
12		
15		
18		
x		

2. Is the relationship between a CFL game's purchase price and the company's revenue quadratic? Explain or show your thinking.

Name: ______ Date: ______ Period: ______

Activity 2 What Price to Charge? (continued)

3. Use the values in the table to plot points that represent the revenue r as a function of the purchase price per game in dollars x.



4. What price would you recommend Kiran's company charge per game? Explain your thinking.

Are you ready for more?

The function that uses the price x (in dollars per game) to determine the number of games purchased $18 - x$ (in thousands) is an example of a *demand function*, and its graph is known among economists. Economists are interested in factors that can affect the demand function, and therefore the price suppliers wish to set.

1. Other than price, what factors might increase the number of games purchased?

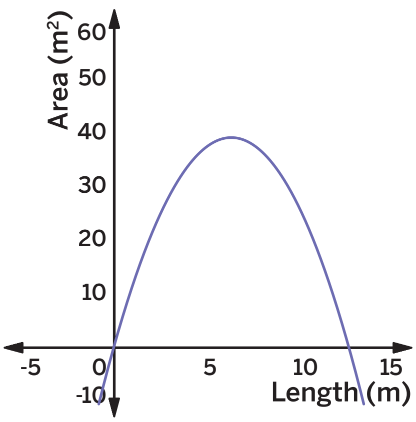
2. If the demand shifted so that you predicted $(20 - x)$ thousand games purchased at a price of x dollars per game, predict what would happen to the price that gives the maximum revenue. Then check your prediction.

Activity 3 Domain, Vertex, and Zeros

As Kiran's business grows, other challenges arise. The following graphs model some of the company's challenges. For each function:

- Describe an appropriate domain. Think about possible upper or lower limits as input values, and whether all numbers make sense as input values. Then describe how the graph could be modified to better show the domain.
- Identify or estimate the vertex and zeros of the function. Describe what each represents in the scenario.

1. Kiran's construction team designs the company's rectangular office space. Its perimeter is 25 m and one side has a length of x. The function $A(x) = x \cdot \frac{25 - 2x}{2}$ models the area, in square meters, of the office space.

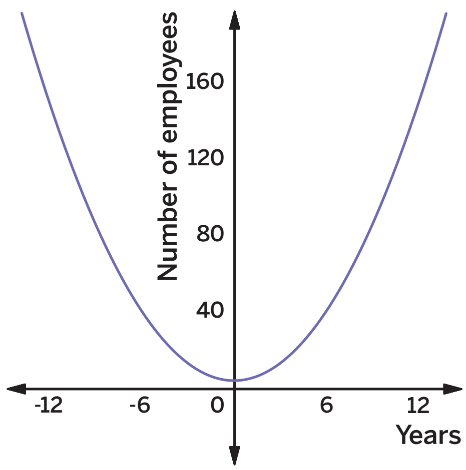


a **Domain:**

b **Vertex:**

c **Zeros:**

2. As Kiran's company grows, so does the number of employees. This is expressed as the function $f(n) = n^2 + 4$, where n is number of years since the company started.



a **Domain:**

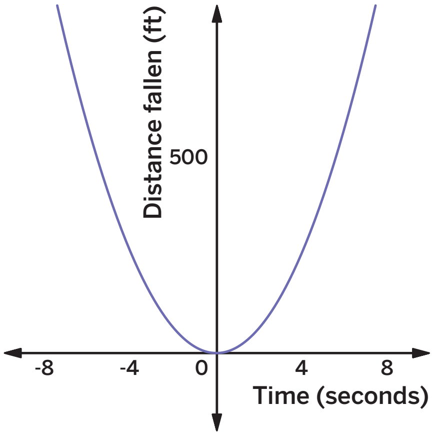
b **Vertex:**

c **Zeros:**

Name: .... Date: .... Period: ....

Activity 3 Domain, Vertex, and Zeros (continued)

3. The company purchases the footage of each game from the league. To make sure they are able to get a close-up shot of any action during a game, the camera operators look at the distance in feet that a football will fall t seconds after being dropped, using the function $g(t) = 16t^2$.

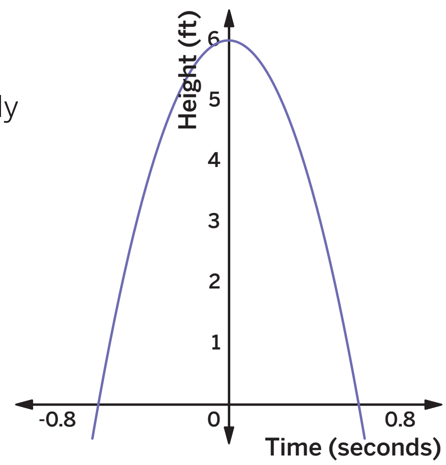


a **Domain:**

b **Vertex:**

c **Zeros:**

4. The company analyzes when customers pause and replay a game and concludes one of the most replayed plays is a fumble, when a player unexpectedly drops the ball. Kiran thinks fumbles are a good time to place an advertiser logo on screen. To help determine how long the logo should be on screen, the company looks at the height of a suddenly dropped football, which is represented by the function $h(t) = 6 - 16t^2$, where t is the number of seconds after being dropped from a height of 6 ft.



a **Domain:**

b **Vertex:**

c **Zeros:**

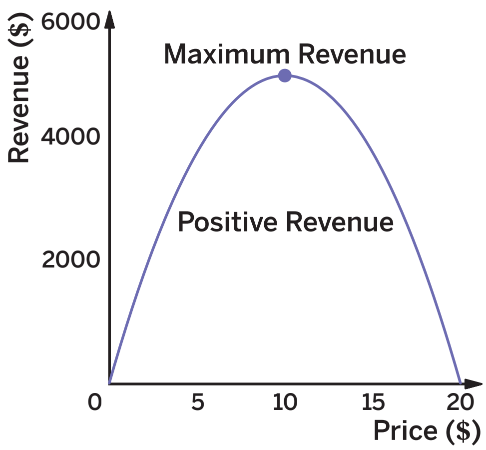
STOP

Summary

In today's lesson . . .

You saw how quadratic functions can be used to study revenue. The term *revenue* means the amount of money collected when a product is sold. Let's study a graph of the revenue of a product.

A company's revenue generally increases with the introduction of a new product. Everyone wants the new product. Eventually the revenue from the product decreases after a certain point, for many reasons, e.g., the price is too high.



For this company and product shown in the graph, the maximum revenue occurs at the vertex: The revenue is $\$5{,}000$ when the product's selling price is $\$10$.

The domain of this function is between $\$0$ and $\$20$, which can be found by identifying the graph's zeros. This domain tells us that if the product is priced above $\$20$, the model predicts that no revenue will be generated.

Reflect:

Name: .... Date: .... Period: ....

Practice

1. Based on past musical productions, a theater predicts the number of tickets they will sell using the expression $400 - 8p$, where each ticket is sold at p dollars.

a Complete the table.

Ticket price (\$)	Number of tickets sold	Revenue (\$)
5		
10		
15		
30		
45		
50		
p		

b For which ticket price(s) will the theater earn no revenue? Explain your thinking.

c The theater needs to earn at least $\$3{,}200$ in revenue to break even (to not lose money). For which ticket price(s) would the theater break even? Explain your thinking.

2. A company sells running shoes. If the price of a pair of shoes in dollars is represented by p, the company estimates that it will sell $50000 - 400p$ pairs of shoes. Write an expression for the revenue, in dollars, earned from selling running shoes priced at p dollars each.

Practice

Name: ________ Date: ________ Period: ________

3. The function f represents the revenue, in dollars, a school can expect to receive if it sells $220 - 12x$ coffee mugs for x dollars each. The graph of the function is shown. Select *all* the statements that describe this scenario.

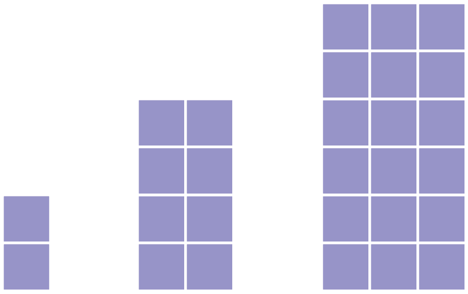
A. At \$2 per coffee mug, the revenue will be \$196.

B. The school expects to sell 160 mugs if the price is \$5 each.

C. The school will earn about \$1,000 if it sells the mugs for \$10 each.

D. The revenue will be at least \$800 if the price is between \$5 and \$13.

4. Write an equation to represent the relationship between the figure number n and the number of unit squares y. Describe how each part of the equation relates to the pattern.



5. A bacteria population p is modeled by the equation $p = 100000 \cdot 2^d$, where d is the number of days since the population was first measured. Select *all* statements that are true in this scenario.

A. The bacteria population 4 days before it was first measured was 6,250.

B. The bacteria population 3 days before it was first measured was 800,000.

C. The population was greater than 1,000,000 one week after it was first measured.

D. $100000 \cdot 2^{-2}$ represents the bacteria population 2 days before it was first measured.

6. Choose any method to multiply the expressions.

a. $x \cdot 5$

b. $x \cdot 5x$

c. $5x(x + 1)$

SUB-UNIT

3 Quadratic Expressions

Narrative Connections

How do you put the “quad-” in quadratics?

The term “quad” shows up in a lot of places. Someone giving birth to four identical twins has *quadruplets*. If you want to go offroading on four big wheels, you would rent a *quad* bike. And if you needed a courtyard with four sides to hang out between classes, you would head straight for the *quadrangle*, or *quad* for short.

Number sleuths out there will notice that the number four keeps showing up wherever the term “quad” appears.

So, where is the four in *quadratics*? If you look at the exponent, we’re always raising terms to the power of *two*, not four.

But the word used for raising to the second power is *squaring*, because that is how you determine a square’s area, given its side length. And because a square has *four* sides, that is where the name *quadratic* comes from! It comes from the Latin *quadrare*, a verb that means “to make something square.”

This relationship between the *side length* and *area* of squares (and other rectangles) is at the very heart of quadratics. Whether it is a falling object under the grip of gravity, or a fledgling business trying to make a profit, by understanding how a rectangle’s side lengths and area are related, we can use them to map out any quadratic relationship.

Unit 5 | Lesson 10

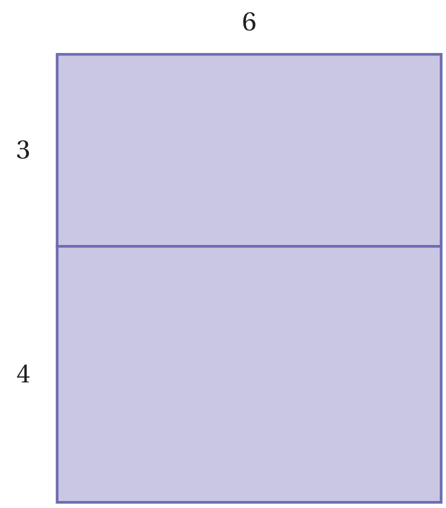
Equivalent Quadratic Expressions (Part 1)

Let's use diagrams to write quadratic expressions.

Warm-up Rectangles

Study the diagram.

Han believes the diagram shows the expression $6 \cdot 3 \cdot 4$. What do you think the diagram shows? Explain or show your thinking.



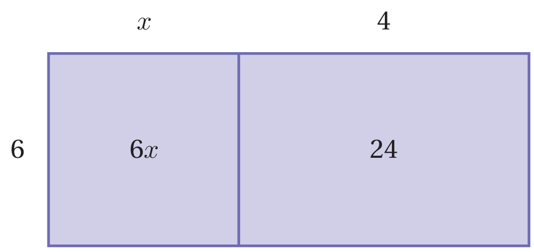
Critique and Correct: Your teacher will share an incorrect argument for what this diagram shows. With your partner, critique and correct the argument, then explain why you corrected it.

Log in to Amplify Math to complete this lesson online.

Name: ____________ Date: ________ Period: ________

Activity 1 What Property Is It?

Ancient Babylonians used area diagrams to write equivalent expressions. For example, the expression $6(x + 4)$ represents the area of the rectangle shown, where the width w is 6 and the length ℓ is $x + 4$. The rectangle is split into two smaller rectangles. The area diagram shows the area of each smaller rectangle. Their sum, $6x + 24$, represents the area of the larger rectangle.



1. Which properties in algebra do you think are related to the Babylonian geometric method for writing equivalent expressions? Explain your thinking.

2. Use the Distributive Property to expand $5(x + 2)$.

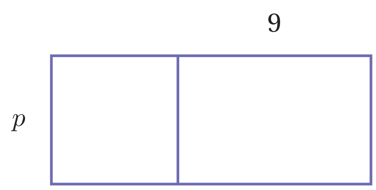
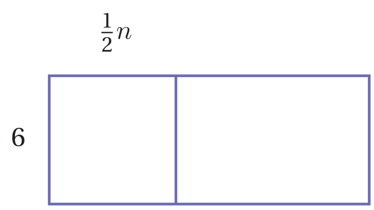
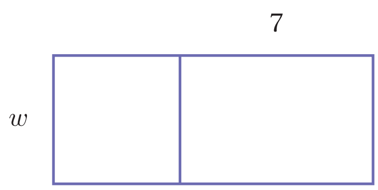
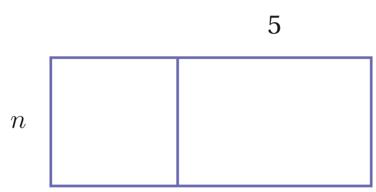
3. Sketch a rectangle to show side lengths of 5 and $(x + 2)$. Then make it an area diagram by showing the areas of each of the smaller rectangles.

4. What does the product of $5(x + 2)$ represent in this context? Explain your thinking.

5. The side lengths of a rectangle are given by the expressions x and $(2x + 1)$. Sketch an area diagram of the rectangle. Then write an expression representing the area of the rectangle.

Activity 2 Determining Area

Each expression represents the area of a rectangle, written as the product of its side lengths. Complete the table.

Expression	Area diagram	Equivalent expression
$p(4p + 9)$		
		$3n + 12$
$(0.5w + 7)w$		
$n(2n + 5)$		

1. How are area diagrams helpful for modeling each expression?

2. How can you use the areas of the two smaller rectangles to determine the area of the larger rectangle?

Name: .... Date: .... Period: ....

Activity 3 I Have . . . Who Has . . . ?

You will play the game "I Have . . . Who Has?" to match products of linear expressions with their respective area diagrams. You will be given playing cards and will need a sheet of paper and a pencil. Please attend carefully to the instructions.

Rules:

- Play begins with the card that says, "This is the first card."
- Whoever has this card reads the "I have the expression ___ times ___. Who has an equivalent expression?" question aloud. Then they write the problem on the board.
- Everyone else draws an area diagram to determine the product.
- Raise your hand if you have the matching area diagram on your card.
- Explain how the area diagram on your card indicates the equivalent expression.
- Then read the question on your card aloud and write the question on the board.
- Repeat until there is one card remaining that says, "This is the last card."

Historical Moment

Babylonian Math Problems
Babylonian mathematicians often worked with squares. They used square shapes and square numbers to perform complex calculations. Can you solve this Babylonian math problem from 1700 BC?

"To find the area of a rectangle, the excess of the length over the width is added, giving 120; moreover, the sum of the length and width is 24. Find the dimensions of the rectangle."

This information tells you two things.

- The sum of the length and width is 24.
- The sum of the area and the difference between the length and width is 120.

What are the length and width of the rectangle? Explain your thinking.

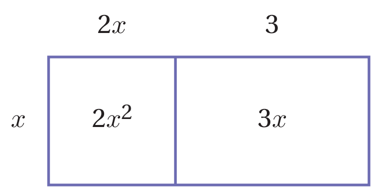
Mathematical textbook containing 247 questions where the area of a field is given and a quadratic equation must be used to determine its dimensions. *Oriental Institute Museum*, University of Chicago. Daderot/CC0

STOP

Summary

In today's lesson . . .

You saw that a quadratic expression can be written in different equivalent forms. You also used quadratic expressions to represent the area of a rectangle, where each linear expression (factor) represents a side length. By decomposing the rectangle into two smaller rectangles, you can determine the areas of the smaller rectangles and find their sum to determine the area of the larger rectangle. You can use an area diagram of a rectangle to visualize the Distributive Property.



$$x(2x + 3) = 2x^2 + 3x$$

Reflect:

Name: .... Date: .... Period: ....

1. Sketch an area diagram to show that $2x\left(3x + \frac{3}{2}\right)$ is equivalent to $6x^2 + 3x$.

2. Select *all* expressions that are equivalent to $x^2 + 4x$.

   A. $x(x + 2)$

   B. $x(x + 4)$

   C. $(x + 4)x$

   D. $4(x + 4)$

   E. $x \cdot x + 4 \cdot x$

3. Tyler drew an area diagram to expand the expression $4x(2x + 5)$.

	$2x$	5
$4x$	$6x^2$	9

   a. Explain Tyler's mistake.

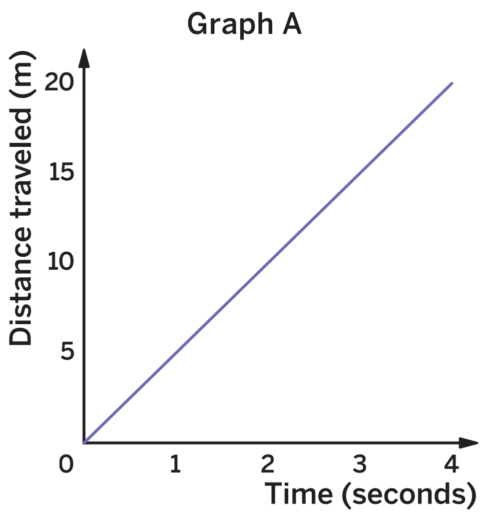
   b. Write the correct equivalent form of the expression $4x(2x + 5)$.

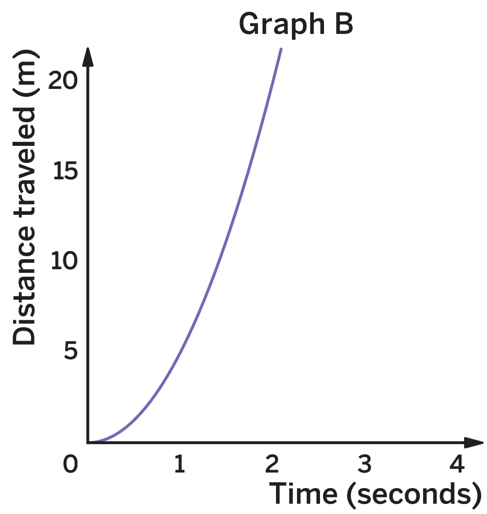
4. Explain or show why the values of the expression 3^x will eventually overtake the values of the expression $3x^2$.

Name: ______ Date: ______ Period: ______

Practice

5. A baseball traveled d meters t seconds after being dropped from the top of a building. The distance traveled by the baseball can be modeled by the function $d(t) = 5t^2$.





Which graph best represents this situation? Explain your thinking.

6. Draw an area diagram modeling the expression $(x + 2)(x + 6)$. Determine the partial products and write an equivalent expression. Show or explain your thinking.

Name: ______ Date: ______ Period: ______

Unit 5 | Lesson 11

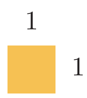
Equivalent Quadratic Expressions (Part 2)

Let's examine how the product of two binomial factors can be expressed as an equivalent quadratic expression.

Warm-up Algebra Tiles

1. Determine the area of each figure.

a

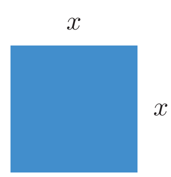


Area = ______

b

Area = ______

c



Area = ______

2. Use the figures in Problem 1 to sketch a model of the given expressions.

a $5x + 10$

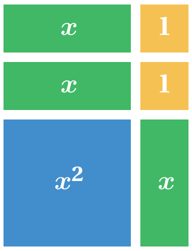
b $x^2 + 2x$

c $x^2 + 7x + 10$

Log in to Amplify Math to complete this lesson online.

Activity 1 Using Tiles to Find Equivalent Quadratic Expressions

Using 1 large square, 3 rectangles, and 2 unit squares, it is possible to build a rectangle, such as the one shown. The area of the entire rectangle can be written as the product $(x + 1)(x + 2)$, or alternatively as the sum $x^2 + 3x + 2$.



Now, it is your turn to build rectangles and write expressions, given the following sets of tiles.

1. Given tiles: 1 large square, 4 rectangles, 4 unit squares

   a. Is it possible to build a rectangle using these given tiles? If so, draw the model.

   b. Write the expression as a product, if possible. Then write the expression as a sum.

2. Given tiles: 1 large square, 5 rectangles, 3 unit squares

   a. Is it possible to build a rectangle using these given tiles? If so, draw the model.

   b. Write the expression as a product, if possible. Then write the expression as a sum.

Name: ______ Date: ______ Period: ______

Activity 1 Using Tiles to Find Equivalent Quadratic Expressions (continued)

3. Given tiles: 1 large square, 3 rectangles, 9 unit squares

a Is it possible to build a rectangle using these given tiles? If so, draw the model.

b Write the expression as a product, if possible. Then write the expression as a sum.

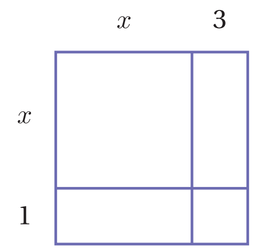
4. Given tiles: 1 large square, 5 rectangles, 6 unit squares

a Is it possible to build a rectangle using these given tiles? If so, draw the model.

b Write the expression as a product, if possible. Then write the expression as a sum.

Activity 2 Using Diagrams to Determine Products

1. Refer to the diagram of a rectangle with side lengths $(x + 1)$ and $(x + 3)$. Show that $(x + 1)(x + 3)$ and $x^2 + 4x + 3$ are equivalent expressions.



2. Draw a diagram for each expression. Use your diagram to write an equivalent expression.

   a. $(x + 9)(x + 6)$

   Equivalent expression:

   b. $(x + 5)^2$

   Equivalent expression:

   c. $(2 + x)(4 + x)$

   Equivalent expression:

   d. $(2x + 1)(x + 3)$

   Equivalent expression:

3. Consider each of the equivalent expressions in Problem 2. What do you notice about the first and last terms?

Name: .... Date: .... Period: ....

Activity 2 Using Diagrams to Determine Products (continued)

4. Each diagram in Problem 2 has *four* boxes, and each equivalent expression has *three* terms. Explain why this happens.

5. How could you find any equivalent quadratic expression *without* the use of diagrams?

6. Write an equivalent expression for $(x + 3)(x + 5)$.

STOP

Summary

In today's lesson . . .

You used algebra tiles to represent the binomial factors of quadratic expressions. You saw that the square tile represents $x \cdot x$ or x^2, the rectangular tile represents $x \cdot 1$ or x, and the unit square represents $1 \cdot 1$ or 1.

You used these tiles to model and write the product of two binomial factors, and wrote equivalent quadratic expressions that represented the product. You also extended your understanding of the Distributive Property, multiplying two binomial linear expressions without using an area diagram or algebra tiles.

In general, when a quadratic expression is written in the form $(x + p)(x + q)$, you can apply the Distributive Property to rewrite the expression as $x^2 + px + qx + pq$ or $x^2 + (p + q)x + pq$.

$$\begin{aligned}
&(x + 2)(x + 3) \\
&= x(x + 3) + 2(x + 3) \\
&= x^2 + 3x + 2x + (2)(3) \\
&= x^2 + (3 + 2)x + (2)(3)
\end{aligned}$$

Reflect:

Practice

Name: ______________ Date: ______________ Period: ______________

1. Sketch a model to show that the expression $(2x + 5)(x + 3)$ is equivalent to the expression $2x^2 + 11x + 15$.

2. Elena drew a diagram to find the product of the expression $(x + 5)(2x + 3)$.

	$2x$	3
x	$2x^2$	$3x$
5	$7x$	8

a Explain Elena's mistake.

b What is the correct product of the expression $(x + 5)(2x + 3)$?

Practice

Name: .... Date: .... Period: ....

For Problems 3–4, refer to the following scenario.

The revenue a band earns is based on the number of tickets sold at past concerts, where the price of a ticket is p dollars. The number of tickets sold is modeled by the expression $600 - 10p$. The table shows the number of concert tickets the band expects to sell and the expected revenue at different ticket prices.

3. Complete the table.

Ticket price ($)	Number of tickets	Revenue ($)
10		
15		
20		
30		
35		
45		
50		
60		
p		

4. In this model, at what ticket price(s) will the band earn no revenue? Explain your thinking.

5. Match each quadratic expression with an equivalent expanded expression.

a $(x + 2)(x + 6)$ .... $x^2 + 12x + 32$

b $(2x + 8)(x + 2)$ .... $2x^2 + 10x + 12$

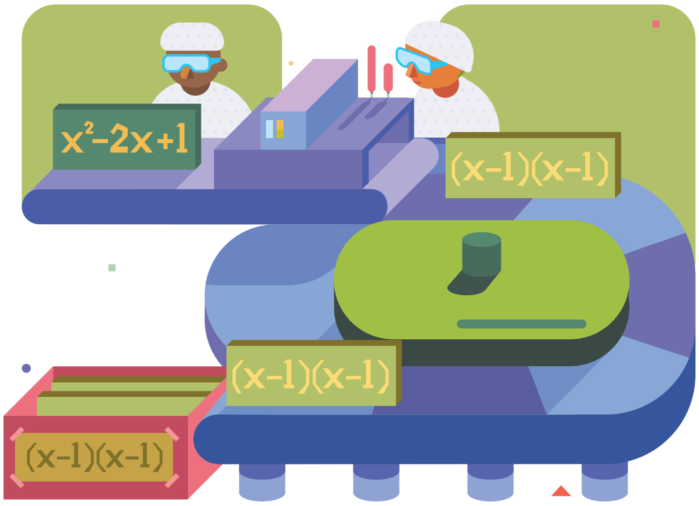
c $(x + 8)(x + 4)$ .... $2x^2 + 12x + 16$

d $(x + 2)(2x + 6)$ .... $x^2 + 8x + 12$

Name: ________ Date: ________ Period: ________

Unit 5 | Lesson 12

Standard Form and Factored Form



Let's write quadratic expressions in different forms.

Warm-up Algebra Talk

What do these equations have in common?

$40 - 8 = 40 + n$

$3 - \frac{1}{2} = 3 + n$

$25 + (-100) = 25 - n$

$72 - n = 72 + 6$

Log in to Amplify Math to complete this lesson online.

Activity 1 Finding Products of Differences

1. Are the two expressions $(x - 1)(x - 1)$ and $x^2 - 2x + 1$ equivalent? Use an area diagram to show your thinking.

2. Draw an area diagram and write an equivalent expression for each given expression.

Expression	Area diagram	Equivalent expression
$(x + 1)(x - 1)$		
$(x - 2)(x + 3)$		
$(x - 2)^2$		
$(2x - 1)(x + 4)$		

Name: .... Date: .... Period: ....

Activity 2 What Is the Standard Form? What Is the Factored Form?

Study the quadratic expressions in each set.

Set A	Set B
$x^2 - 1$	$(2x + 3)x$
$x^2 + 9x$	$(x + 1)(x - 1)$
$\frac{1}{2}x^2$	$3(x - 2)^2 + 1$
$4x^2 - 2x + 5$	$-4(x^2 + x) + 7$
$-3x^2 - x + 4x + 6$	$(x + 8)(-x + 5)$
$1 - x^2$	$x(x - 3)$

1. What are some characteristics of the quadratic expressions in Set A?

2. Both $(x + 1)(x - 1)$ and $(2x + 3)x$ are quadratic expressions written in *factored form*. Why do you think that form is called factored form?

3. Which other expressions in Set B are written in factored form?

Are you ready for more?

Which quadratic expression can be described as being in both standard form and factored form? Explain your thinking.

A. $x(2x + 4)$

B. x^2

C. $3(x^2 - 5) + 1$

Activity 3 I Have . . . Who Has . . . ?

You will play the game "I Have . . . Who Has?" and match quadratic expressions in factored form with equivalent expressions in standard form. You will be given playing cards and will need a blank sheet of paper and a pencil. Please attend carefully to the instructions.

Rules:

- Play begins with the card that says, "This is the first card."
- Whoever has this card reads the "I have ______, who has the equivalent expression?" question aloud. They then write the expression on the board.
- Everyone else writes the equivalent expression.
- Raise your hand if you have the equivalent expression in standard form on the top of your card.
- Explain how the standard form on your card is the equivalent expression.
- Read the bottom of your card aloud and write the expression on the board.
- Repeat until there is one card remaining that says, "This is the last card."

STOP

Name: ________________ Date: ________ Period: ________

Summary

In today's lesson . . .

You worked with two forms of quadratic expressions. The ***factored form*** of a quadratic expression is written as the product of two linear factors. This product can be expanded using the Distributive Property, resulting in standard form. The ***standard form*** of a quadratic expression is given by $ax^2 + bx + c$, where a is the non-zero coefficient of the squared term, b is the coefficient of the linear term, and c is the constant term.

An example of converting a quadratic expression from factored form to standard form is shown.

Begin with the factored form:	$(x + 2)(x + 1)$
Use the Distributive Property:	$= x \cdot x + x \cdot 1 + 2 \cdot x + 2 \cdot 1$
Simplify to write in standard form:	$= x^2 + 3x + 2$

Reflect:

Practice

Name: .... Date: .... Period: ....

1. Write each quadratic expression in standard form. Draw a diagram if needed.

   a $(x + 4)(x - 1)$

   b $(2x - 1)(3x - 1)$

2. Consider the expression $8 - 6x + x^2$.

   a Is the expression in standard form? Explain your thinking.

   b Is the expression equivalent to $(x - 4)(x - 2)$? Explain your thinking.

3. Which quadratic expression is written in standard form?

   A. $(x + 3)x$

   B. $(x + 4)^2$

   C. $-x^2 - 5x + 7$

   D. $x^2 + 2(x + 3)$

4. Jada drops her sunglasses from a bridge. Which equation best represents y, the distance fallen in feet, as a function of time t, in seconds?

   A. $y = 16t^2$

   B. $y = 48t$

   C. $y = 180 - 48t$

   D. $y = 180 - 16t^2$

Name: ______ Date: ______ Period: ______

Practice

5. *Technology required*. Two rocks are launched vertically in the air.

- The height of Rock A is given by the function $f(t) = 4 + 30t - 16t^2$.
- The height of Rock B is given by function $g(t) = 5 + 20t - 16t^2$.

In both functions, t represents the time, in seconds, and the height is measured in feet.

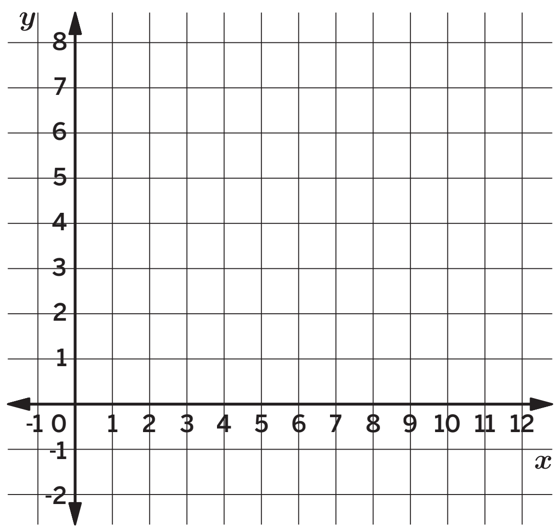
a. What is the maximum height of each rock?

b. Which rock reaches its maximum height first? Explain or show your thinking.

6. Graph the function $f(x) = -\frac{2}{3}x + 6$. Then answer each question about your graph.

a. How is the value $-\frac{2}{3}$ from the function represented on the graph?

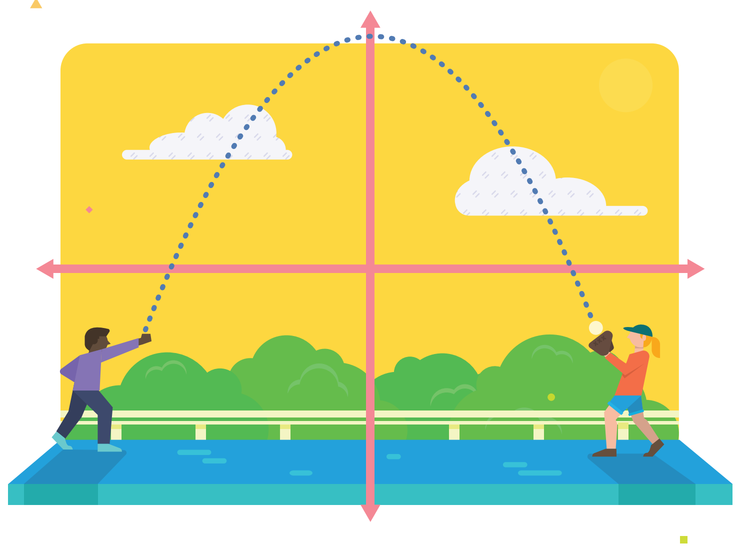
b. How is the value 6 from the function represented on the graph?



Unit 5 | Lesson 13

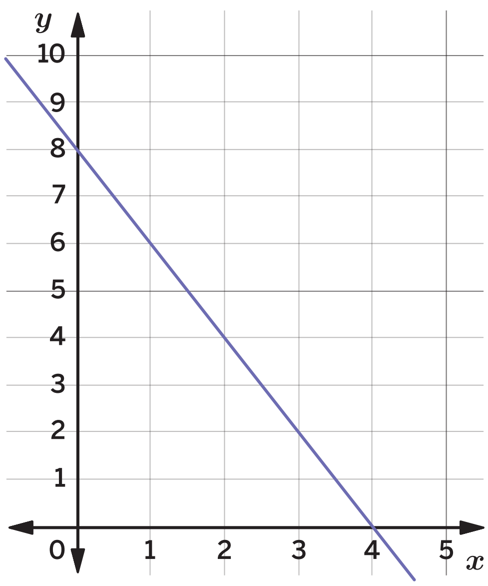
Graphs of Functions in Standard and Factored Forms

Let's explore what information each quadratic form reveals about the properties of their graphs.



Warm-up A Linear Equation and Its Graph

Relate each term in the equation $y = 8 - 2x$ to the graph and state the x-intercept.



Log in to Amplify Math to complete this lesson online.

Name: ______ Date: ______ Period: ______

Activity 1 Playing Catch

Kiran and Andre play catch. They toss a ball back and forth until it hits the ground. The height, in feet, of Kiran's toss is modeled by the function $h(t) = 6 + 29t - 16t^2$, where time t is measured in seconds. The ball is tossed at an initial height of 6 ft and an initial vertical speed of about 29 ft/second.

Three Reads:
You will read this introduction three times to help make sense of the scenario. Your teacher will tell you what to look for during each read.

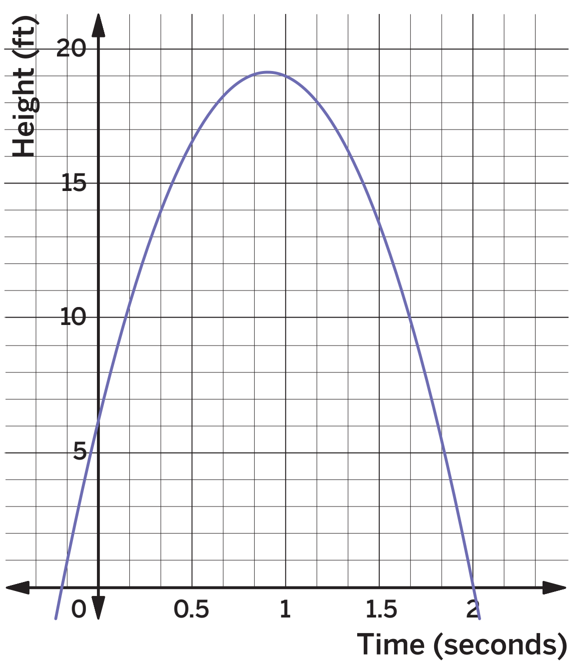
1. Is the function $h(t) = 6 + 29t - 16t^2$ written in standard form? Explain or show your thinking.

2. Does the function $g(t) = (-16t - 3)(t - 2)$ also define Kiran's toss, written in factored form? Explain or show your thinking.

3. The graphs of $g(t)$ and $h(t)$ are shown.

a Identify or approximate the x- and y-intercepts.

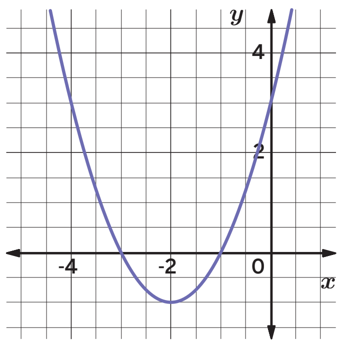
b What do each of these points represent in this scenario?



Activity 2 Relating Functions and Their Graphs

Identify the x-intercepts and y-intercept of each function's graph.

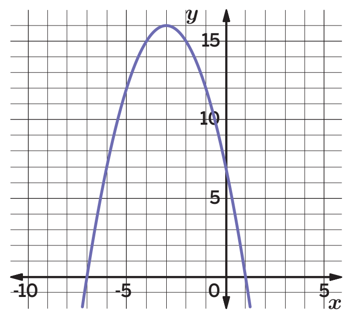
1. $f(x) = (x + 3)(x + 1)$



x-intercepts:

y-intercept:

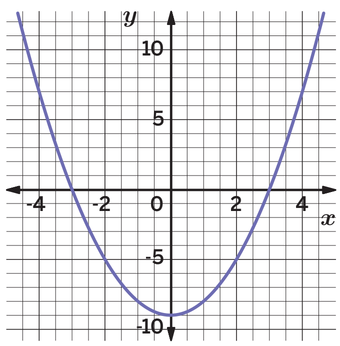
2. $g(x) = -x^2 - 6x + 7$



x-intercepts:

y-intercept:

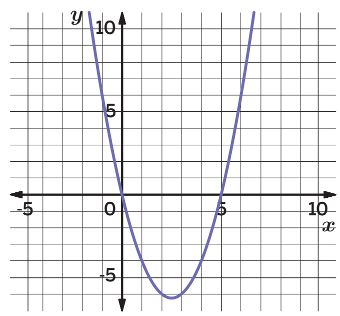
3. $h(x) = x^2 - 9$



x-intercepts:

y-intercept:

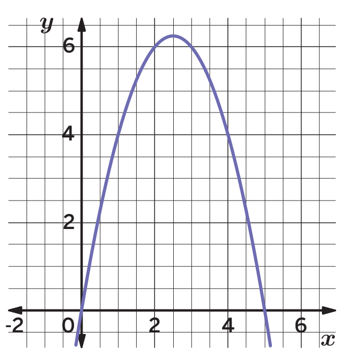
4. $i(x) = x(x - 5)$



x-intercepts:

y-intercept:

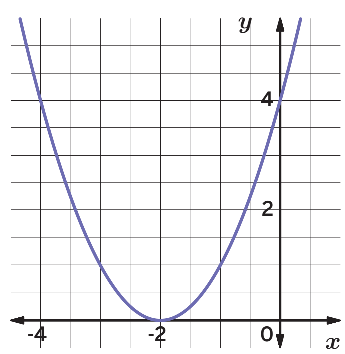
5. $j(x) = x(5 - x)$



x-intercepts:

y-intercept:

6. $k(x) = x^2 + 4x + 4$



x-intercepts:

y-intercept:

Name: ______ Date: ______ Period: ______

Activity 2 Relating Functions and Their Graphs (continued)

7. What do you notice about the x-intercepts, the y-intercepts, and the constant terms in the factored and standard forms defining each function?

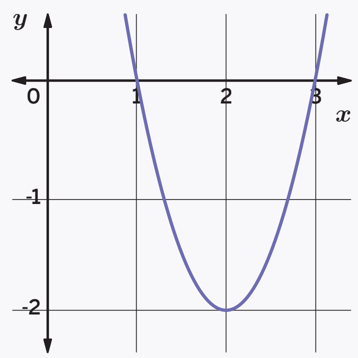
8. Consider the function $p(x) = (x - 9)(x - 1)$. What do you think are the x- and y-intercepts of the graph that represents this function?

9. Which quadratic form, factored or standard, best helps identify the x-intercepts? The y-intercept?

Reflect: How did you use your strengths to complete the activity?

Are you ready for more?

Study the graph and determine the values of a, p, and q that will make $y = a(x - p)(x - q)$ the equation represented by the graph.



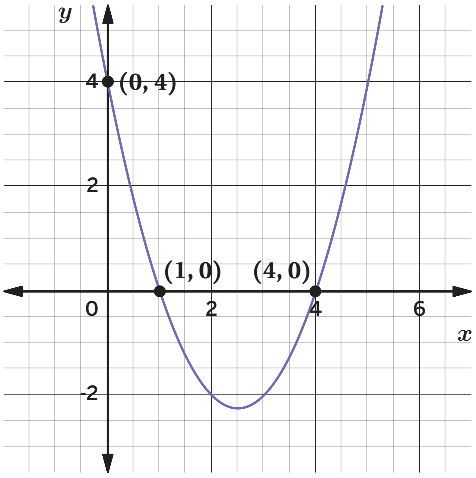
STOP

Summary

In today's lesson . . .

You saw that different forms of quadratic functions can provide information about their graphs. A quadratic function expressed in factored form can tell you about the x-intercepts of its graph.

For example, $f(x) = (x - 4)(x - 1)$, has x-intercepts of 1 and 4 because they are located at $(1, 0)$ and $(4, 0)$. The x-intercepts of the graph are the zeros of the function (the input values that produce an output of 0). Meanwhile, a quadratic function in standard form tells you the y-intercept of the function's graph. For example, the graph representing $f(x) = x^2 - 5x + 4$ has a y-intercept at $(0, 4)$.



The shape of the graph of a quadratic equation or function is called a *parabola*.

Reflect:

Name: ________ Date: ________ Period: ________

Practice

1. Consider the quadratic function $f(x) = (x - 7)(x + 3)$.

   a. Without graphing, identify the x-intercepts of the function's graph Explain your thinking.

   b. Expand $(x - 7)(x + 3)$ and use the expanded form to identify the y-intercept of the graph of the function. Explain your thinking.

2. Where are the x-intercepts located on the graph of the function $g(x) = (x - 2)(x + 1)$?

   A. $(2, 0)$ and $(-1, 0)$

   B. $(2, 0)$ and $(1, 0)$

   C. $(-2, 0)$ and $(1, 0)$

   D. $(-2, 0)$ and $(-1, 0)$

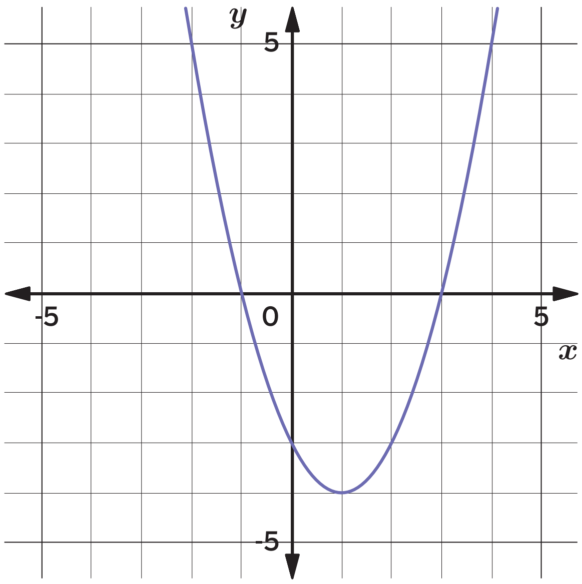
3. The graph of a quadratic function is shown. Which of the following could define this function?

   A. $h(x) = (x + 3)(x + 1)$

   B. $h(x) = (x + 3)(x - 1)$

   C. $h(x) = (x - 3)(x + 1)$

   D. $h(x) = (x - 3)(x - 1)$



Practice

Name: ________________ Date: ________ Period: ________

4. Write each quadratic expression in standard form. Draw a diagram to explain your thinking.

a. $(x - 3)(x - 6)$

b. $(x - 4)^2$

c. $(2x + 3)(x - 4)$

d. $(4x - 1)(3x - 7)$

5. A company sells video games, where p represents the price of each game in dollars. The company estimates it will sell $20000 - 500p$ games. Which expression represents the revenue, in dollars, if the company sells the estimated number of games?

A. $(20000 - 500p) + p$

B. $(20000 - 500p) - p$

C. $(20000 - 500p) \div p$

D. $(20000 - 500p) \cdot p$

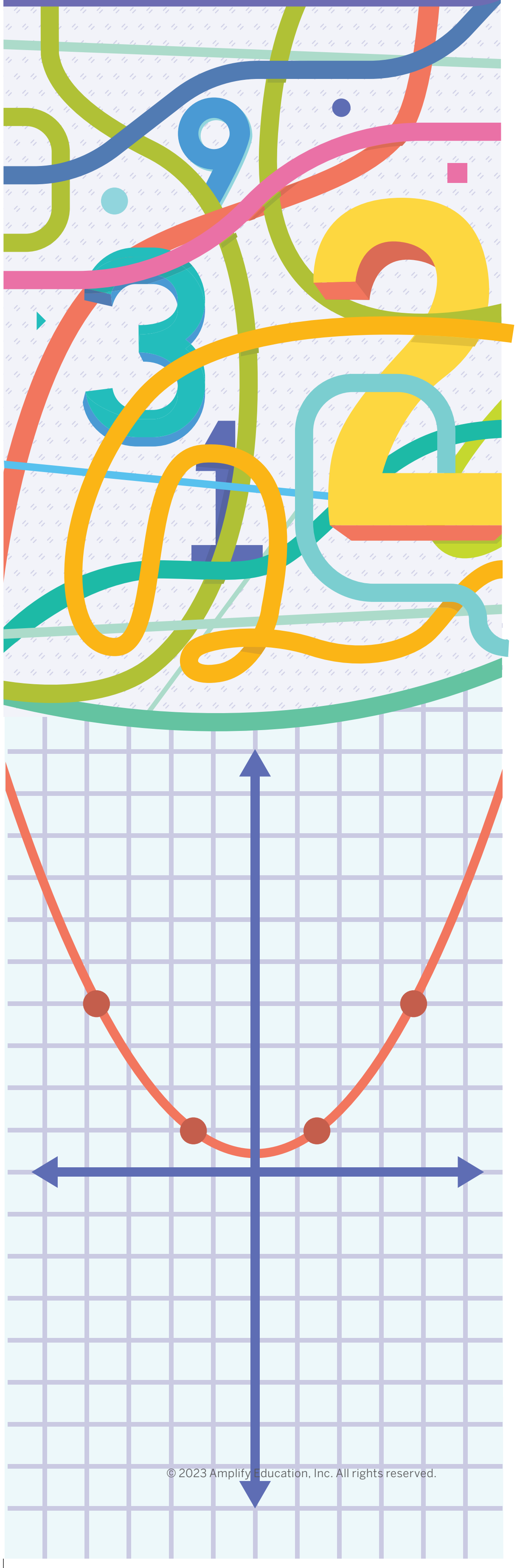
6. Consider the function, $f(x) = x^2 - 5x + 4$.

a. What is the y-intercept of the graph of the function?

b. An equivalent way of writing this function is $f(x) = (x - 4)(x - 1)$. What are the x-intercepts of this function's graph?

SUB-UNIT

4 Features of Graphs of Quadratic Functions



Narrative Connections

Mirror, mirror on the wall, what's the fairest function of them all?

You know the old saying, "Beauty is only skin deep"? Well, for quadratics, beauty goes a little deeper than that.

It can be challenging to see how elegant a quadratic truly is. At first glance, a quadratic can look like a mess of numbers, variables, and operators — possessing all the grace of a jumble of tangled power cords.

But once a quadratic function is graphed, then, all at once, the beauty that has been hiding under those coefficients and exponents becomes immediately clear.

It is always a wonderfully symmetric, swooping curve.

With this, we can see things we could not see before: the way the curve opens — either upward, or downward, how shallow or deep its rise and fall, and the exact spot where it pivots. Every point on that curve is a nugget of rich information. And when its independent variable is time, then it serves as a prediction for the future, a record of the past, and a picture of what is possible.

Previously, we looked at the different forms quadratic functions could take. Now, let's see how we can use these forms as a blueprint for sculpting such swift, elegant figures.

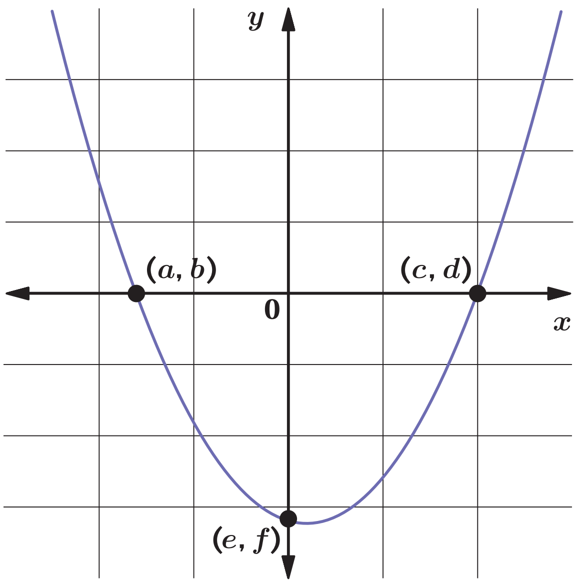
Unit 5 | Lesson 14

Graphing Quadratics Using Points of Symmetry

Let's graph some quadratic functions using factored form.

Warm-up Finding Coordinates

The following graph shows the function $w(x) = (x + 1.6)(x - 2)$. Three points on the graph are labeled.



Determine the values of a, b, c, d, e, and f. Be prepared to explain your thinking.

Log in to Amplify Math to complete this lesson online.

Name: ______ Date: ______ Period: ______

Activity 1 Comparing Two Graphs

1. Complete the table of values for each function. Determine the x-intercepts and the location of the vertex in each function's graph. Be prepared to explain your thinking.

x	$f(x) = x(x+4)$	$g(x) = x(x-4)$
-5	5	45
-4		
-3		
-2	-4	12
-1	-3	5
0		
1		
2		
3		-3
4	32	
5	45	

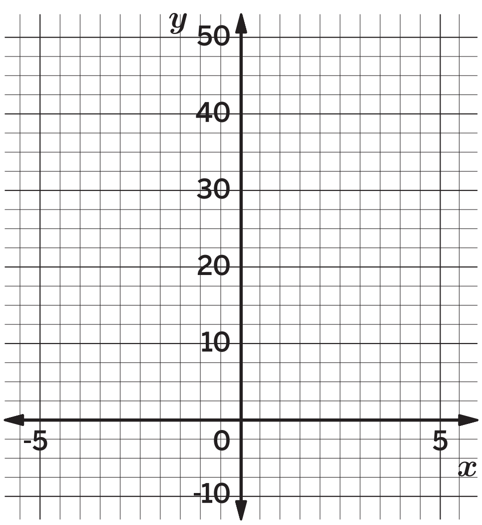
x-intercepts of $f(x)$:

x-intercepts of $g(x)$:

Vertex of $f(x)$:

Vertex of $g(x)$:

2. Plot the points from the tables on the same coordinate plane. Use dots to represent $f(x)$ and x's to represent $g(x)$.



Activity 1 Comparing Two Graphs (continued)

3. What do you notice about the two graphs you drew?

Are you ready for more?

Consider the functions $p(x) = (x - 1)(x - 3)$ and $q(x) = (x + 1)(x + 3)$. Complete the following problems without using graphing technology.

1. What are the x-intercepts of the graphs of $p(x)$ and $q(x)$?

2. What is the x-coordinate of each graph's vertex?

3. How would you find the y-coordinate of the vertex for each graph?

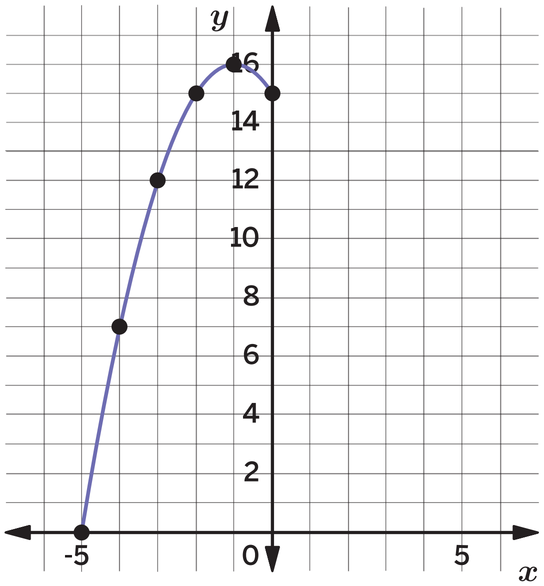
4. The y-intercept of both graphs is $(0, 3)$. What is another point on each graph with a y-coordinate of 3?

Name: ________________ Date: ________ Period: ________

Activity 2 The Axis of Symmetry

Use the partial table and partial graph of the quadratic function shown as you complete this activity.

x	$f(x)$
–3	12
–2	15
–1	16
0	15
1	
2	
3	



1. What are the coordinates of the vertex of the function?

2. Use a straightedge to draw a vertical line through the x-coordinate of the vertex. This line is called the quadratic function's ***axis of symmetry***.

   a. What is the equation of this line? Label it on the graph.

   b. Why do you think this line is called the *axis of symmetry*?

   c. How might you use the axis of symmetry to help you sketch the rest of the parabola?

3. Complete the table and plot the corresponding points on the graph.

4. How does the vertex relate to the equation of the axis of symmetry? Will this always be the case? Explain your thinking.

Activity 2 The Axis of Symmetry (continued)

5. What are the x-intercepts? How do they relate to the axis of symmetry?

6. Use the x-intercepts to write the function $f(x)$ in factored form.

7. Write the function $f(x)$ in standard form.

8. For a function written in standard form, the equation of the axis of symmetry can be found by using the equation $x = -\frac{b}{2a}$. Verify that this equation works for the function $f(x)$.

Are you ready for more?

Lin made a mistake trying to find the axis of symmetry for the function $h(x) = x^2 - 2x - 2$. Her graph also has an error. Her work is shown.

1. What is the correct axis of symmetry for this function? Show your thinking.

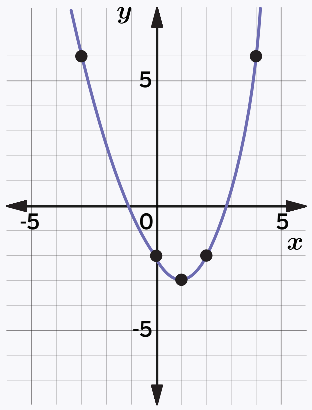
2. What mistake did Lin make when finding the axis of symmetry?

3. By looking at the graph, how could Lin have determined that she made a mistake? How can she fix it?

Lin's work:

The axis of symmetry is $x = -1$ because

$$x = -\frac{b}{2a} = -\frac{2}{2(1)} = -1.$$



Name: ________________ Date: ________ Period: ________

Activity 3 What Do We Need to Sketch a Graph?

1. The functions $f(x)$, $g(x)$, and $h(x)$ are defined in the following table. Without graphing, determine the x-intercepts, the x-coordinate of the vertex, and the equation of the axis of symmetry for each function.

Function	x-intercepts	x-coordinate of the vertex	Axis of symmetry
$f(x) = (x + 3)(x - 5)$			
$g(x) = 2x(x - 3)$			
$h(x) = (x + 4)(4 - x)$			

2. Use graphing technology to graph the functions $f(x)$, $g(x)$, and $h(x)$. Use the graphs to verify your responses to Problem 1.

3. Sketch a graph that represents the function $k(x) = (x - 7)(x + 11)$. How could you find the y-coordinate of the vertex? Be prepared to explain your thinking.

4. The axis of symmetry of the function $k(x)$ in Problem 3 is $x = -2$. How can you use this information to verify that you graphed the function correctly?

STOP

Summary

In today's lesson . . .

You saw that writing a quadratic function in factored form, like $f(x) = (x + 1)(x - 3)$, is helpful for determining the zeros of the function — the x-intercepts of its graph, where $f(x) = 0$.

Factored form can also help you determine a quadratic function's vertex, where a function reaches its least or greatest value. Because a quadratic function is symmetric, the x-coordinate of its vertex is located halfway between the two x-intercepts. If you evaluate $f(x)$ at this value of x, you can also determine the y-coordinate of the vertex.

The x-coordinate of the vertex also provides the function's ***axis of symmetry***, the vertical line that divides the parabola into two symmetric halves. The formula for the axis of symmetry is $x = -\frac{b}{2a}$, which can be used to determine the x-coordinate of the vertex when you do not have a graph. You can also use the axis of symmetry to visually verify that your parabola is symmetric.

Reflect:

Name: ________________ Date: ________ Period: ________

1. Select *all* the true statements about the graph that represents the equation $y = 2x(x - 11)$.

A. Its x-intercepts are located at $(-2, 0)$ and $(11, 0)$.

B. Its x-intercepts are located at $(0, 0)$ and $(11, 0)$.

C. Its x-intercepts are located at $(2, 0)$ and $(-11, 0)$.

D. It has exactly one x-intercept.

E. The x-coordinate of its vertex is -4.5.

F. The x-coordinate of its vertex is 11.

G. The x-coordinate of its vertex is 4.5.

H. The x-coordinate of its vertex is 5.5.

2. Select *all* the equations whose graphs have a vertex with an x-coordinate of 2.

A. $y = (x - 3)(x - 4)$

B. $y = (x - 2)(x + 2)$

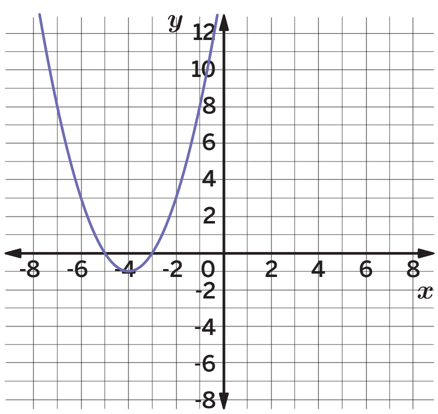
C. $y = (x - 1)(x - 3)$

D. $y = x(x + 4)$

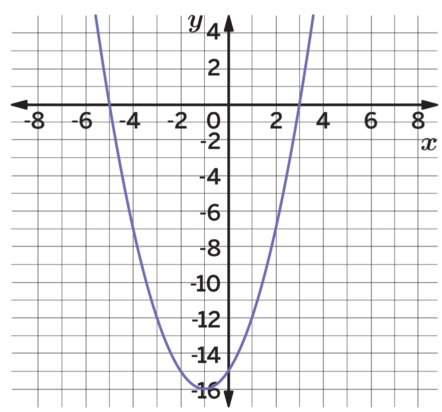
E. $y = x(x - 4)$

3. Which is the graph of the equation $y = (x - 3)(x + 5)$?

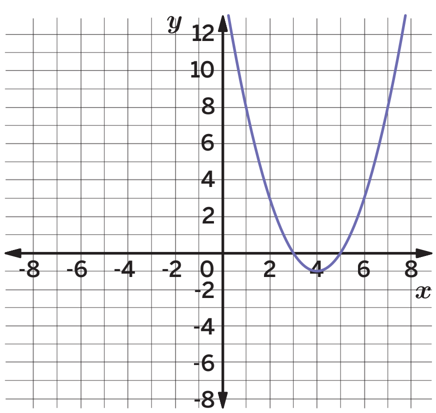
A.



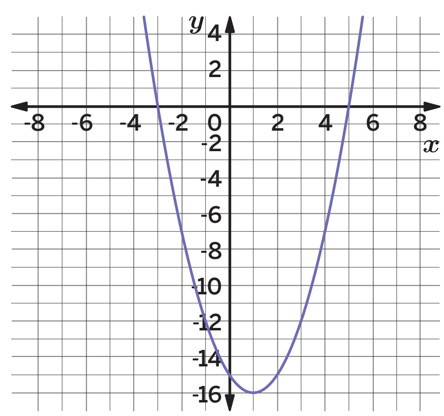
C.



B.



D.



Practice

Name: .... Date: .... Period: ....

4. Is the expression $(s + t)^2$ equivalent to the expression $s^2 + 2st + t^2$? Show or explain your thinking.

5. Suppose $g(x)$ takes a student's grade as its input, and gives a student's name as its output. Explain why $g(x)$ is *not* a function.

6. Determine the x-intercepts, the x-coordinate of the vertex of the graph, and the equation of the axis of symmetry for each equation.

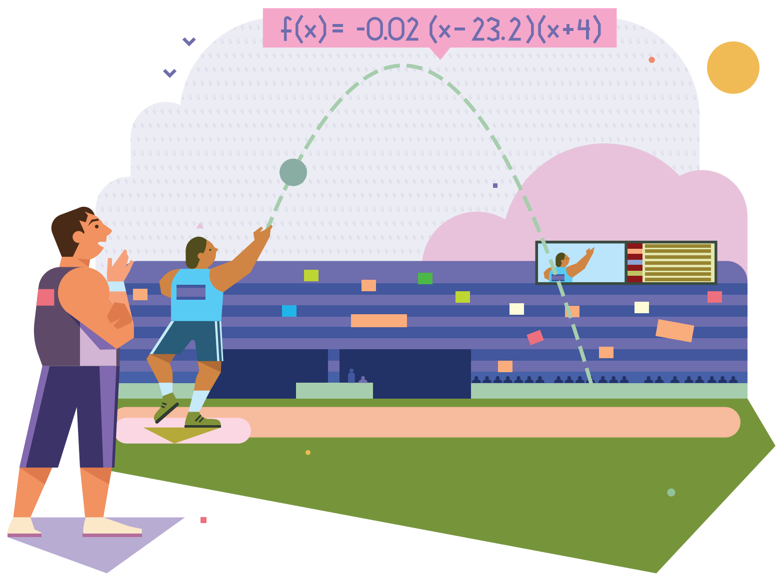
Equation	x-intercepts	x-coordinate of the vertex	Axis of symmetry
$y = x(x - 2)$			
$y = (x - 4)(x + 5)$			
$y = -5x(3 - x)$			

Name: ______ Date: ______ Period: ______

Unit 5 | Lesson 15

Interpreting Quadratics in Factored Form

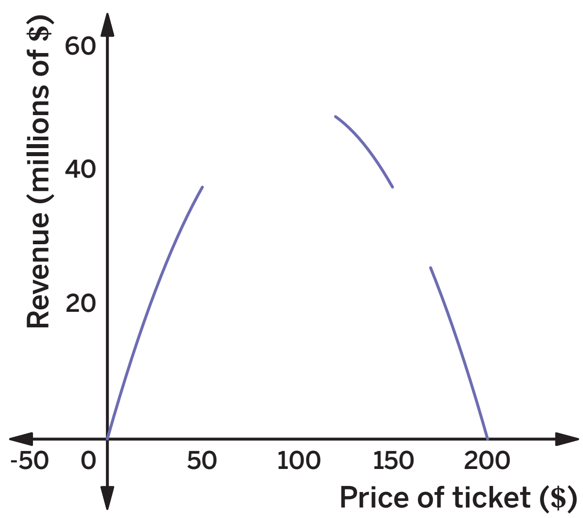
Let's see what information the factored form provides, given a context.



Warm-up The Revenue From Tickets

The top professional soccer teams in the world generate hundreds of millions of dollars in revenue every year, some of which comes from ticket sales. The owner of the KickStars United team wants to determine the maximum revenue the team made last year from ticket sales, but some data is missing. She is presented with a graph of the available data.

What information can be gathered about the revenue from tickets?



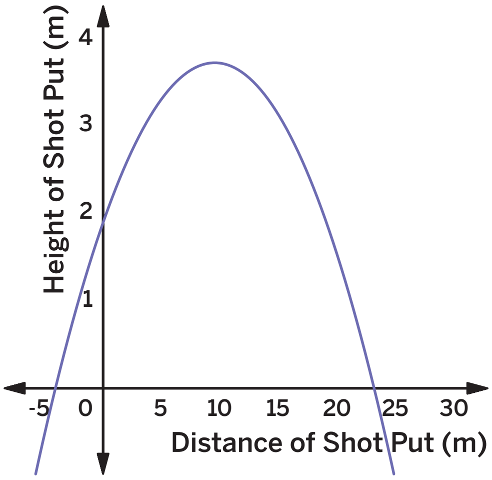
Log in to Amplify Math to complete this lesson online.

Activity 1 The Shot Put World Record

The male shot put world record was broken by Randy Barnes of the United States in 1990, with a throw of 23.12 m.

After years of training, Noah is attempting to break Barnes' record. Noah's coach films his throw from the moment he releases the shot (the heavy metal ball), and models the shot's path using a computer.

Study the graph, which shows the trajectory of the shot for Noah's throw. Then solve the following problems.



1. Does Noah's throw break the world record? Explain your thinking.

2. Noah's throw is modeled by the function, $f(x) = -0.02(x - 23.2)(x + 4)$. Use this function to verify or support your response to Problem 1.

3. Analyze the graph. What other information can be learned about Noah's throw? Support your thinking using points from the graph.

4. What domain is most appropriate to model Noah's throw? Explain your thinking.

Are you ready for more?

A different computer program models the path of the shot put with the function $g(x) = -0.02x^2 + 0.384x + 1.856$, where $g(x)$ is the height of Noah's throw when the shot has traveled x meters horizontally. Which function, $f(x)$ or $g(x)$, better models the throw? Explain your thinking.

Name: ________ Date: ________ Period: ________

Activity 2 The Perfect Synchronized Dive

Synchronized diving became an Olympic sport in 2000. In this sport, two people dive together from a platform, mirroring each other's twists, turns, and movements along the way. Importantly, the divers must reach the water at the same time and distance from the platform.

katatonia82/Shutterstock.com

Andre and Tyler are synchronized divers, jumping from a platform that is 10 m high. Andre and Tyler successfully mirror each other's moves, but they often have trouble landing at the same distance from the platform.

1. The function $f(x) = -\frac{8}{3}(x + 1.5)(x - 2.5)$ represents the distance, in meters, each diver is above the pool, as a function of their distance from the platform x, which is also measured in meters. The vertex of the function is located at $\left(0.5, \frac{32}{3}\right)$.

a How many meters from the platform will they enter the water? Explain your thinking.

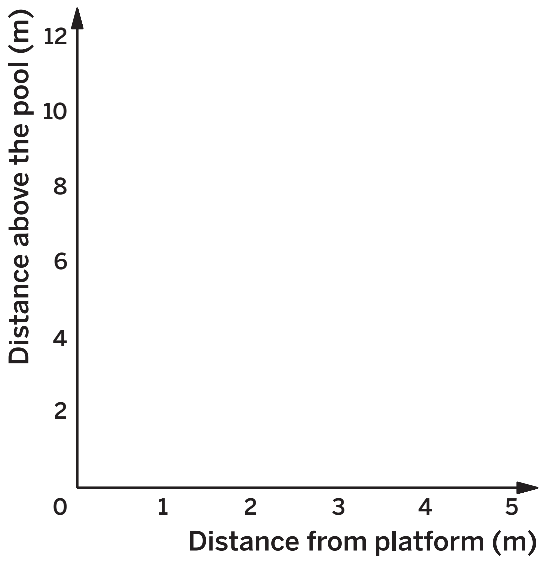
b Is the entire graph of $f(x)$ used to represent their perfect dive? Explain or show your thinking.

Activity 2 The Perfect Synchronized Dive (continued)

2. Andre is perfecting a jump that follows the path of $f(x)$. Tyler is still adjusting, and thinks changing his jump from the platform will allow him to match Andre perfectly. Tyler wants to jump along a path that is modeled by the function $g(x) = -\frac{7}{3}(x + 1.5)(x - 2.5)$.

a Will Tyler be able to match Andre with this jump? Explain your thinking.

b Use graphing technology to help you sketch both $f(x)$ and $g(x)$, to verify your response to part a.



3. Using graphing technology, try writing an alternative function for Tyler, so that his path perfectly matches Andre's, but the function's other zero is not -1.5, as it was for $f(x)$.

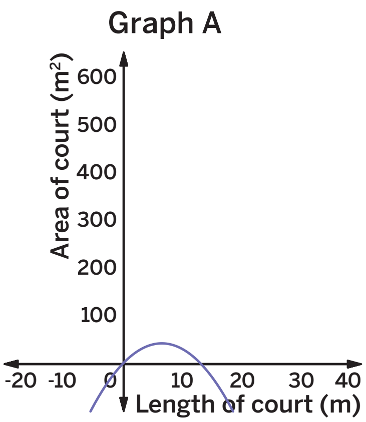
Name: ________ Date: ________ Period: ________

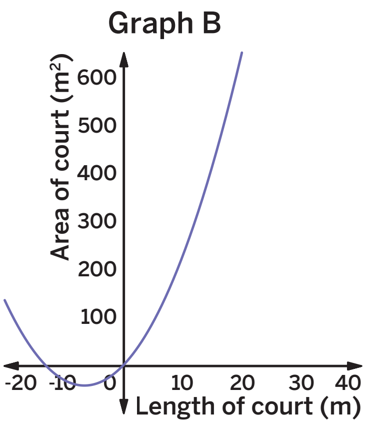
Activity 3 A Wheelchair Basketball Court

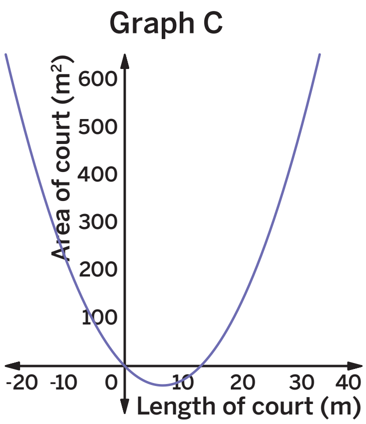
Clare and Kiran are teammates on a wheelchair basketball team. They create a design for the construction of a basketball court at their local park.

1. The width of the court should be 13 m less than its length.

a Which graph best represents the area of the court as a function of its length? Explain your thinking. (Hint: Write an equation to represent the scenario.)







b What are the coordinates of the vertex? Is it meaningful in this context? Explain your thinking.

Activity 3 A Wheelchair Basketball Court (continued)

2. Clare and Kiran estimate that they need *at least* $300\ m^2$ to have enough room to play. The area of the park where they will build their court is $600\ m^2$.

a What are the approximate least and greatest lengths that can be used for the court?

b Revise and sketch the graph you selected in Problem 1 to reflect these limitations.

3. An official wheelchair basketball court has a length of 28 m and width of 15 m. Plot a point on the graph in Problem 2 that represents this court size.

4. Clare and Kiran created a court using the greatest length from Problem 2. How would their court be different from an official court? Explain your thinking.

Reflect: What was the goal of the activity? Explain how you met the goal.

STOP

Name: Date: Period:

Summary

In today's lesson . . .

You saw that the key features of quadratic functions you have learned about so far — the x- and y-intercepts, the vertex, and whether the graph opens upward or downward — can help you solve real-world problems. You can use these features to help sketch the shape of the graph, and to better understand the relationship between the two variables in context.

In many scenarios, only the positive x-intercepts are meaningful. For a given context, the domain for the scenario might even exclude the intercepts or the vertex.

Also, while you can efficiently determine the x-coordinate of the vertex of a quadratic function in factored form, determining the y-coordinate requires more work. By substituting the x-coordinate of the vertex back into the function, you can determine its y-coordinate.

Reflect:

Practice

Name: .... Date: .... Period: ....

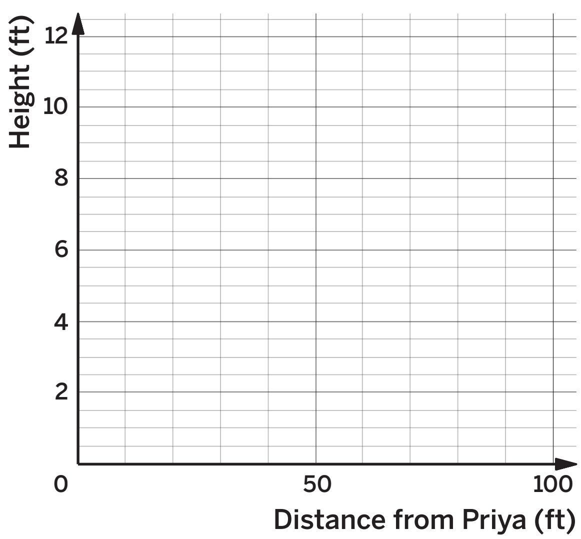
1. A golfer begins their first round of golf. The first hole is located 150 yd away from the tee (the starting point). The function $f(x) = -0.002x(x - 200)$ represents the ball's height as a function of the distance x from the golfer. All distances are measured in yards.

   a. Does the function predict that the golfer will hit the ball too long or too short? Explain your thinking.

   b. Assuming the golfer hits the ball in the direction of the hole, how far away from the hole is the ball predicted to land? Explain your thinking.

2. In the final seconds of a basketball game, Priya has the ball underneath her own net. She throws the ball to her teammate at the opposite end of the court, 90 ft away. The function $h(x) = -0.004(x - 90)(x + 15)$ represents the height of Priya's pass as a function of the distance x from Priya. All distances are measured in feet.

   a. Sketch a graph of Priya's pass.



b Elena, the tallest player of the opposing team, stands 37.5 ft away from Priya. Elena's hands can reach a height of 10 ft when she jumps in the air. Will Elena be able to block Priya's pass and ruin her last-minute full court throw? Explain your thinking.

3. Tyler is shopping for a truck. He finds two trucks that he likes: a red truck that costs $\$7{,}200$, and a slightly older truck that costs 15% less than the red one. What is the price of the older truck?

4. Suppose the function f takes a school's class period number as its input and gives the subject of Mai's Friday class as its output. Use function notation to represent the statement: Mai has Algebra class on Friday during 5th period.

5. Write a linear equation for each description.

a $y = 3x$ after it has been shifted up by 2 units.

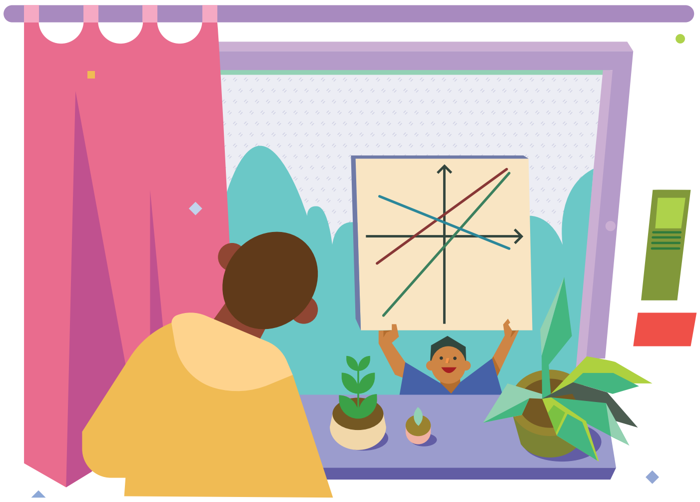
b A line that is steeper than $y = \frac{1}{3}x$.

c A line that is less steep than $y = 3x$

d $y = -2x + 1$ after it has been shifted down by 4 units.

Unit 5 | Lesson 16

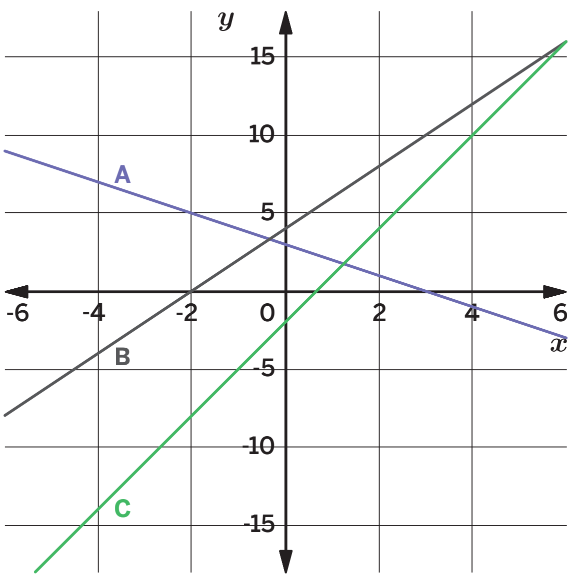
Graphing With the Standard Form (Part 1)



Let's see how changing the values of the coefficients in quadratic functions affects their graphs.

Warm-up Matching Graphs and Linear Equations

Graphs A, B, and C represent three linear equations: $y = 2x + 4$, $y = 3 - x$, and $y = 3x - 2$. Which graph corresponds to which equation? Explain or show your thinking.



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Name: .... Date: .... Period: ....

Activity 1 Changing Quadratics

1. Complete the table to show the values of $g(x) = x^2 + 10$ and $h(x) = x^2 - 3$ for different values of x.

x	-3	-2	-1	0	1	2	3
$f(x) = x^2$	9	4	1	0	1	4	9
$g(x) = x^2 + 10$							
$h(x) = x^2 - 3$							

Use your graphing tool to observe how adding or subtracting a constant term to x^2 affects the graph. Use the values of $g(x)$ and $h(x)$ to describe how the graph of $f(x) = x^2$ changes when a constant term is added or subtracted.

2. Complete the table to show the values of $j(x) = 2x^2$, $k(x) = \frac{1}{2}x^2$, and $p(x) = -2x^2$ for different values of x.

x	-3	-2	-1	0	1	2	3
$f(x) = x^2$	9	4	1	0	1	4	9
$j(x) = 2x^2$							
$k(x) = \frac{1}{2}x^2$							
$p(x) = -2x^2$							

Use your graphing tool to observe how multiplying x^2 by different coefficients affects the graph. Use the values of $j(x)$, $k(x)$, and $p(x)$ to describe how the graph of $f(x) = x^2$ changes when multiplied by a coefficient greater than 1, less than -1, or between -1 and 1.

Activity 2 Quadratic Graphs Galore

Using graphing technology, graph $y = x^2$, and then experiment with each of the following changes to the function. Record your observations.

What happens to the graph of $y = x^2$ when you:

1. Add different constant terms? ($x^2 + 5$, $x^2 + 10$, $x^2 - 3$, etc.)

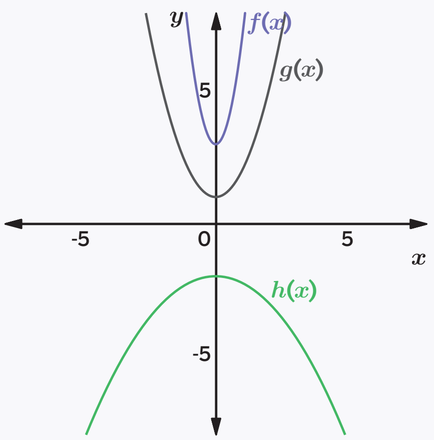
2. Multiply by coefficients greater than 1? ($3x^2$, $7.5x^2$, etc.)

3. Multiply by coefficients less than –1? ($-x^2$, $-4x^2$, etc.)

4. Multiply by coefficients between –1 and 1? ($\frac{1}{2}x^2$, $-0.25x^2$, etc.)

Are you ready for more?

The graph shows the quadratic functions $f(x)$, $g(x)$, and $h(x)$. What can you determine about the coefficients of the squared variable term in each function? Can you identify the coefficients? How do they compare?



Name: .... Date: .... Period: ....

Activity 3 Card Sort: Representations of Quadratic Functions

You and your partner will receive a set of cards. Each card contains either a graph or an equation.

- Take turns with your partner sorting the cards into sets so that each set contains two equations and one graph that represent the same quadratic function.
- For each set of cards that you place together, explain your thinking to your partner.
- For each set of cards that your partner places together, listen carefully to their explanation. If you disagree, discuss your thinking and work to reach an agreement.
- Once all the cards are sorted and discussed, record the equivalent equations, sketch the corresponding graph, and explain why the representations were grouped together.

Equation sets	Graph

STOP

Summary

In today's lesson . . .

You worked with graphs of quadratic functions. A parabola "opens upward" when the vertex is the lowest point on the graph (a minimum), and "opens downward" when the vertex is the highest point on the graph (a maximum). The coefficient of each term of the function written in standard form, $f(x) = ax^2 + bx + c$, provides information about the graph that represents it.

When the coefficient of the squared term a is positive, larger values make the graph steeper (and narrower). Values of a that are closer to 0 make the graph shallower (and wider). When a is negative, the parabola opens downward.

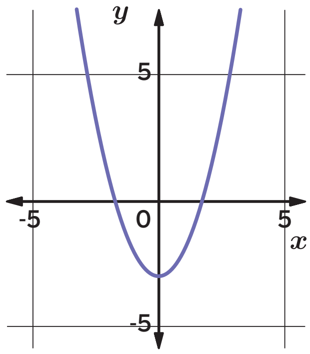
The constant term c tells you about the vertical position of the graph. A function with no constant term — in other words, when $c = 0$ — has a y-intercept at the origin.

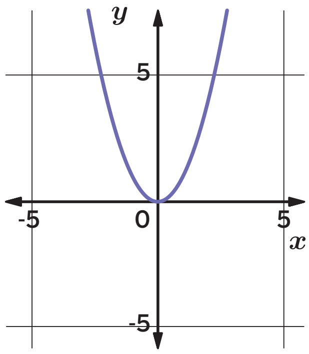
Reflect:

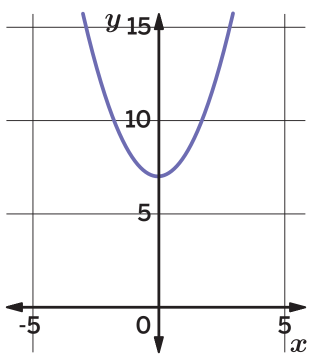
Name: .... Date: .... Period: ....

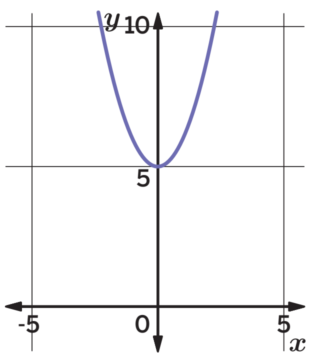
Practice

1. Match each graph with the equation that it represents.

a


b


c


d


.... $y = x^2$

.... $y = x^2 + 5$

.... $y = x^2 - 3$

.... $y = x^2 + 7$

2. The two equations $y = (x + 2)(x + 3)$ and $y = x^2 + 5x + 6$ are equivalent.

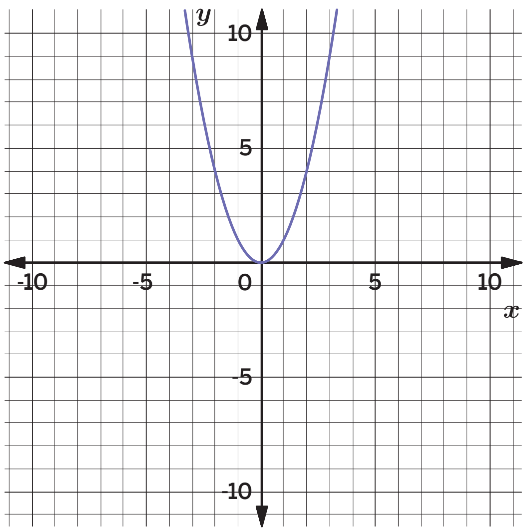
a. Which equation would help you determine the x-intercepts of the graph more efficiently?

b. Which equation would help you determine the y-intercept of the graph more efficiently?

3. Refer to the graph of the equation $y = x^2$. On the same coordinate plane, sketch and label the graph that represents each equation.

a. $y = x^2 - 4$

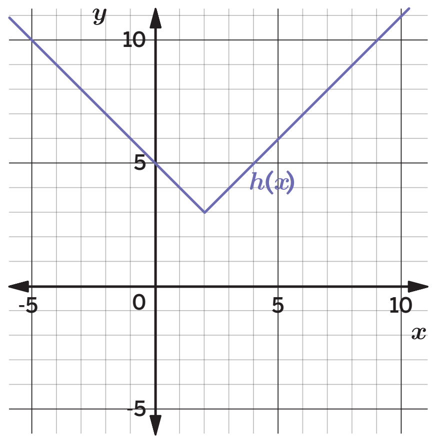
b. $y = -x^2 + 5$



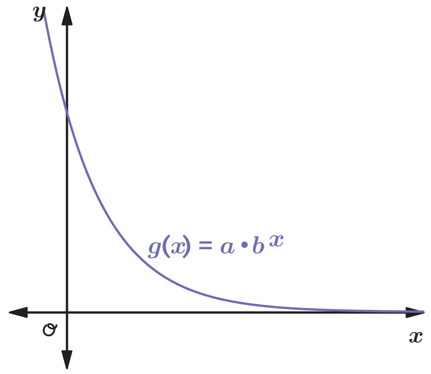
Practice

Name: .... Date: .... Period: ....

4. Describe how the graph of $f(x) = |x|$ has to be shifted to match the graph of $h(x)$ shown. Then write an equation for $h(x)$.



5. Refer to the graph of the function $g(x) = a \cdot b^x$. What can you say about the value of b? Explain your thinking.



6. Determine the x-intercepts, vertex, and y-intercept of the graph of each equation.

Equation	x-intercepts	Vertex	y-intercept
$y = (x - 5)(x - 3)$			
$y = 2x(8 - x)$			

Name: ________________ Date: ________ Period: ________

Unit 5 | Lesson 17

Graphing With the Standard Form (Part 2)

Let's see how changing other values in quadratic functions affects their graphs.

Warm-up Equivalent Expressions

1. Complete each row with an equivalent expression in either standard form or factored form.

Standard form	Factored form
x^2	
	$x(x+9)$
x^2-18x	
	$x(6-x)$
$-x^2+10x$	
	$-x(x+2.75)$

2. Other than what form they are written in, what do the expressions in each column have in common? Be prepared to share your observations.

Log in to Amplify Math to complete this lesson online.

Activity 1 What About the Linear Term?

1. Use graphing technology to explore how changing the linear term of a quadratic function affects its graph.

a Graph the equation $y = x^2$, and then experiment with adding different linear terms (for example, $x^2 + 4x$, $x^2 + 20x$, $x^2 - 50x$). Record your observations.

b Graph the equation $y = -x^2$, and then experiment with adding different linear terms. Record your observations.

2. Use your observations to help you complete the table *without* graphing the equations.

Equation	Factored form	x-intercepts	x-coordinate of vertex
$y = x^2 + 6x$			
$y = x^2 - 10x$			
$y = -x^2 + 50x$			
$y = -x^2 - 36x$			

3. Some quadratic equations have no linear terms. Determine the x-intercepts, if any exist, and the x-coordinate of the vertex of the graph representing each equation. Try graphing the equations to help with your thinking.

a $y = x^2 - 25$

b $y = x^2 + 16$

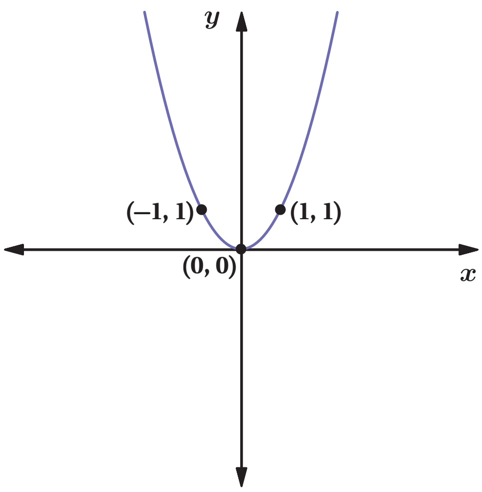
Name: ______ Date: ______ Period: ______

Activity 2 Writing Equations to Match Graphs

Usually you are given an equation and asked to graph it. But sometimes you are given a graph and asked to find the equation. Scientists, like NASA's Katherine Johnson, are able to do both — and not just for quadratics.

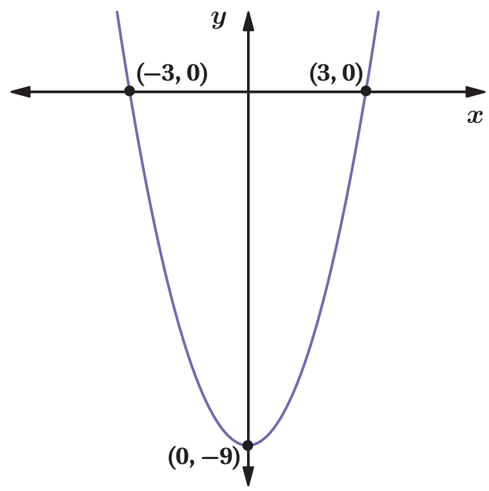
Use graphing technology to help you write the equation that is represented by each graph. Make sure your graph passes through all three points shown. (In Johnson's case, graphing technology did not yet exist. She did her calculations by hand, and was known as a human computer!)

a



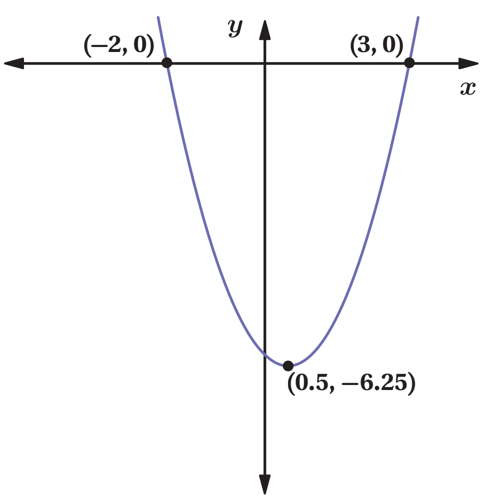
Equation:

b



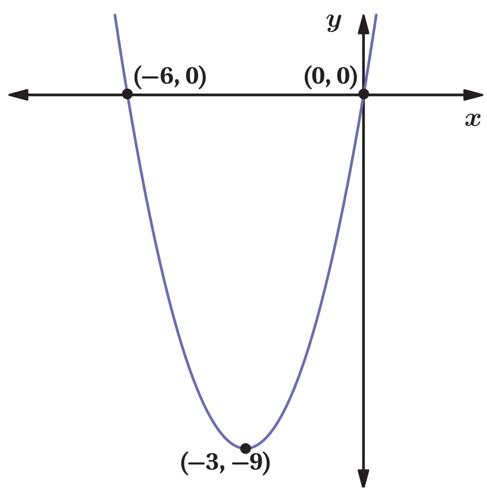
Equation:

c



Equation:

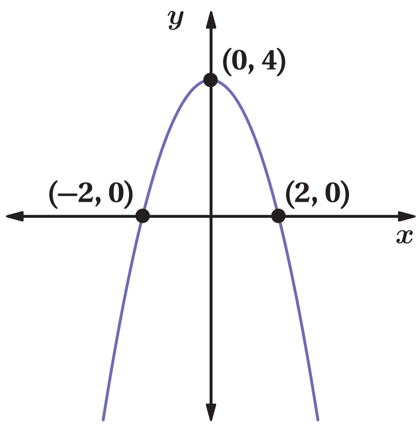
d



Equation:

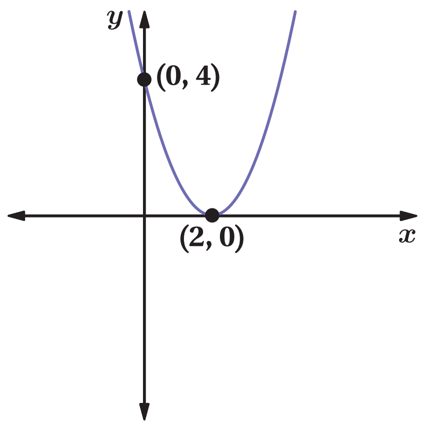
Activity 2 Writing Equations to Match Graphs (continued)

e



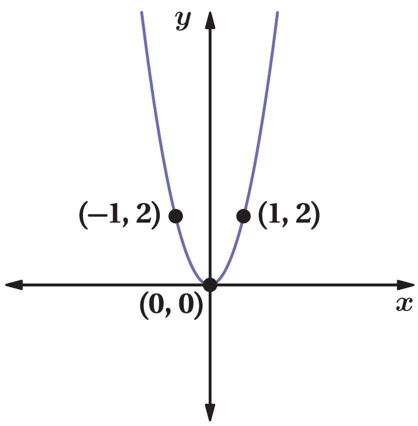
Equation:

f



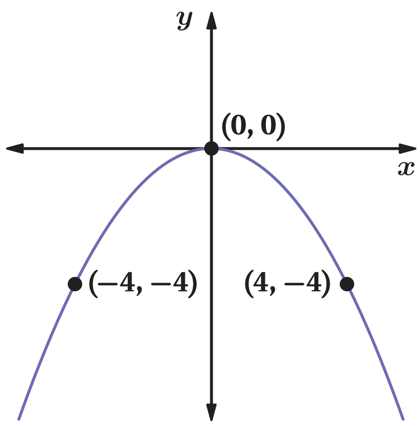
Equation:

g



Equation:

h



Equation:

Featured Mathematician

Katherine Johnson

Katherine Johnson was born in West Virginia in 1918 and was best known for her 35-year career with NASA. With her advanced knowledge of algebra and geometry, she began as a human computer, calculating safe trajectories for some of the first flights into space, despite working in segregated conditions. Later, she became a pioneer in using digital computers to perform orbital calculations. In 2015, in honor of her lifetime of work and achievement, she received the Presidential Medal of Freedom.

STOP

NASA , Bob Nye

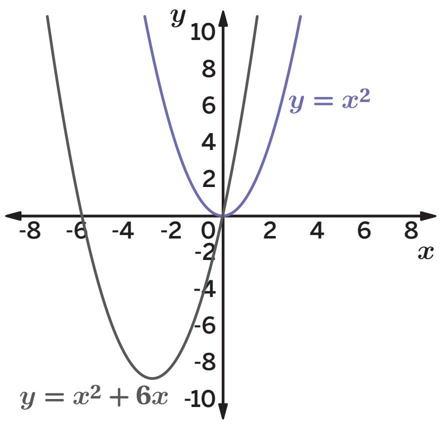
Name: ____________ Date: ________ Period: ________

Summary

In today's lesson . . .

You experimented with changing the values of the linear term bx in the standard form of a quadratic function. You saw how changing b shifts the graph both horizontally *and* vertically.

Recall that the graph representing $y = x^2$ is parabola with a vertex at $(0, 0)$ that opens upwards. When bx is added to x^2, where $b \neq 0$, the graph of $y = x^2 + bx$ is no longer centered on the y-axis. In factored form, $x^2 + bx$ is $x(x + b)$, which means that 0 and $-b$ are the x-intercepts of the equation. The vertex will be located halfway between them, with an x-coordinate of $-\frac{b}{2a}$. Because $a = 1$, the x-coordinate of the vertex in this case is $-\frac{b}{2}$.

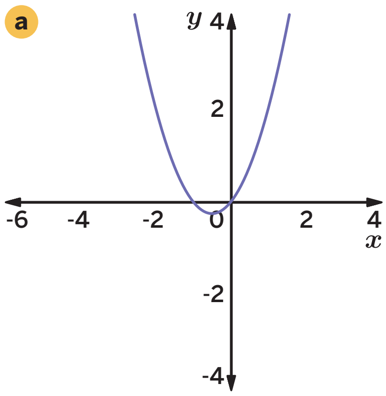


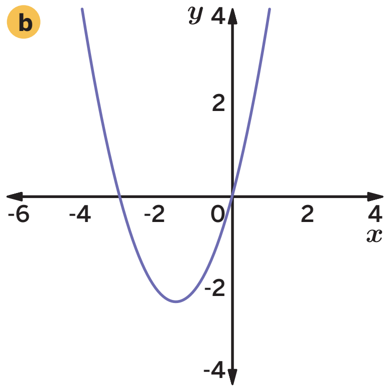
The graphs of $y = x^2$ and $y = x^2 + 6x$ are shown. Notice that they are the same graph, but $y = x^2 + 6x$ is shifted left and down, and its vertex has an x-coordinate of -3.

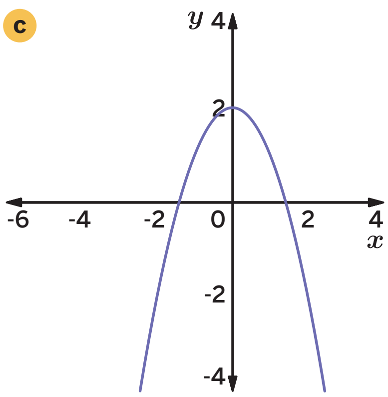
Reflect:

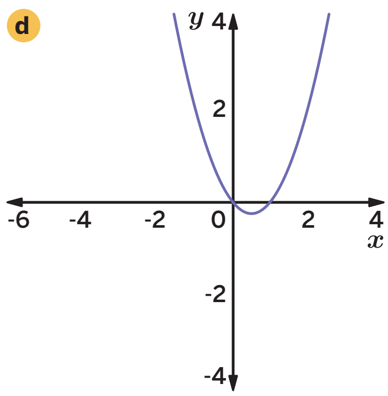
Name: .... Date: .... Period: ....

1. Match each graph with the equation it represents.

a


b


c


d


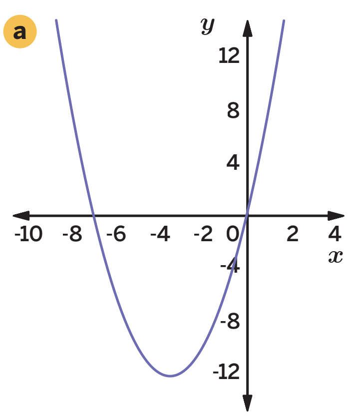
.... $y = x^2 + x$

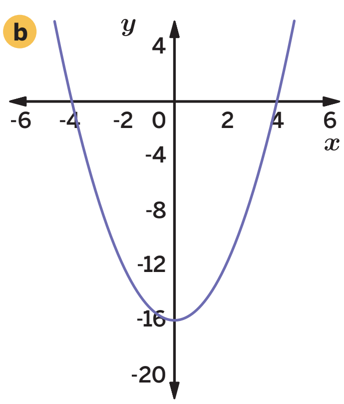
.... $y = -x^2 + 2$

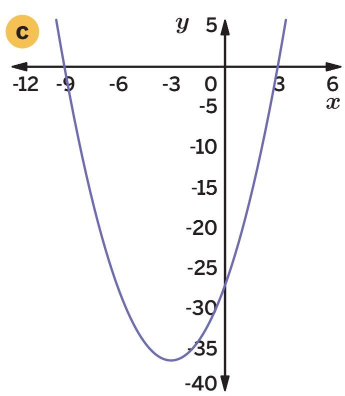
.... $y = x^2 - x$

.... $y = x^2 + 3x$

2. *Technology required*. Write an equation that can be represented by each of the following graphs. Then use graphing technology to check your work.

a


b


c


Name: .... Date: .... Period: ....

Practice

3. Select *all* equations whose graphs open upward.

A. $y = -x^2 + 9x$

B. $y = 10x - 5x^2$

C. $y = (2x - 1)^2$

D. $y = (1 - x)(2 + x)$

E. $y = x^2 - 8x - 7$

4. Match each factored expression with an equivalent expression in standard form.

Factored form	Standard form
a $(x + 3)(x + 4)$	.... $x^2 + 10x + 21$
b $(x + 3)(x + 7)$	.... $3x^2 + 13x + 12$
c $(3x + 4)(x + 3)$	.... $3x^2 + 22x + 7$
d $(x + 7)(3x + 1)$	.... $x^2 + 7x + 12$

5. A bank loans $\$4{,}000$ to a customer at a 9.5% annual interest rate. Write an expression to represent how much the customer will owe, in dollars, after 5 years without payment.

6. Consider the equation $y = (x - 2)(x - 4)$.

a What are the x-intercepts of the graph of this equation?

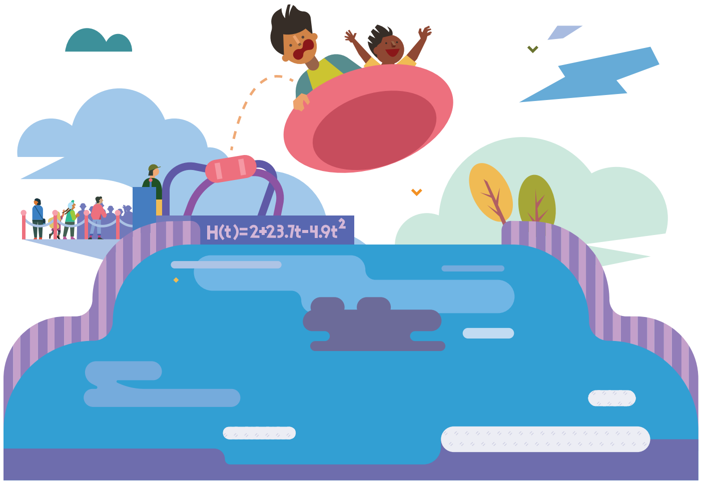
b Find the coordinates of another point on the graph.

c Sketch a graph of the equation.

Unit 5 | Lesson 18

Graphs That Represent Scenarios

Let's examine graphs that represent the paths of objects being launched in the air.



Warm-up A Table Tennis Ball Bounce

The height, in inches, of a single bounce of a table tennis ball is modeled by the function $h(t) = 60t - 75t^2$ where the time t is measured in seconds.

1. Calculate $h(0)$ and $h(0.8)$. What do these values mean in this scenario?

2. When did the ball reach its maximum height? Explain or show your thinking.

Log in to Amplify Math to complete this lesson online.

Name: ____________ Date: ____________ Period: ____________

Activity 1 The Water Catapult

The most popular ride at Wacky Water World is the Catapult. Riders sit in a chair attached to two large elastic ropes that, when released, launch them into the air where they eventually descend into a large pool of water.

1. The function $h(t) = 2 + 23.7t - 4.9t^2$ represents the height of a rider that is launched up in the air as a function of time t, in seconds. The height is measured in meters above ground. The rider is launched with an initial vertical speed of 23.7 m/second.

a What does the term 2 in the equation tell you about this scenario? What about the term $-4.9t^2$?

b If you graph the equation, will the graph open upward or downward? How do you know?

2. Graph the equation using graphing technology. Sketch the graph. Include h- and t-intercepts, as well as the vertex and an appropriate domain.

Activity 1 The Water Catapult (continued)

3. Identify the following features of the graph in Problem 2. Explain what each point means in this scenario.

   a. **h-intercept:**

   b. **t-intercept:**

   c. **Vertex:**

   d. **Domain:**

Are you ready for more?

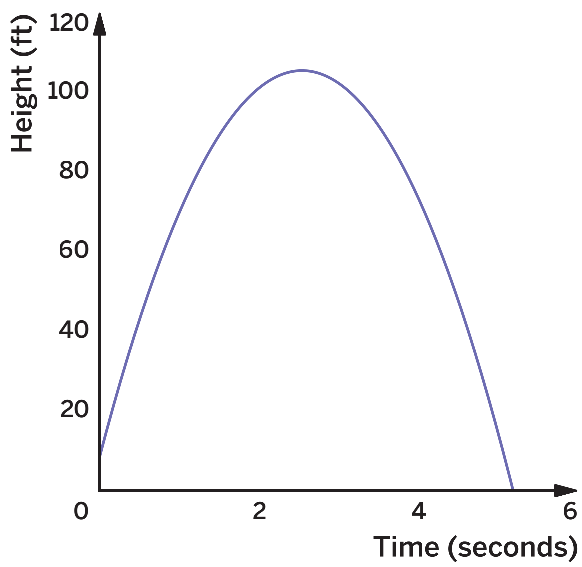
At what approximate initial vertical speed would riders need to be launched in order for them to stay in the air for about 10 seconds? (Assume that they are still launched at an initial height of 2 m and that the effect of gravity pulling them down is the same.)

Name: ____________ Date: ________ Period: ________

Activity 2 Flight of Two Baseballs

The graph represents the height h, in feet, of a baseball as a function of time t, in seconds, after it was hit by Player A.

The function $g(t) = -16\left(t+\frac{1}{16}\right)(t-4)$ also represents the height, in feet, of a baseball t seconds after it was hit by Player B. Without graphing $g(t)$, complete these problems.



1. Which player's baseball stayed in flight longer? Explain your thinking.

2. Which player's baseball reached a greater maximum height? Explain your thinking.

3. How can you determine the height at which each baseball was hit? Explain your thinking.

Activity 3 Info Gap: Rocket Math

Plan ahead: Why will it be important to control your impulses while working with a partner during this activity?

You will be given either a problem card or a data card. Do not show or read your card to your partner.

If you are given the *data* card:	If you are given the *problem* card:
1. Silently read the information on your card. **2.** Ask your partner, "What specific information do you need?" and wait for your partner to ask for information. Only give information that is on your card. (Do not perform any calculations for your partner!) **3.** Before telling your partner the information, ask, "Why do you need to know (that piece of information)?" **4.** Read the problem card, and solve the problem independently. **5.** Share the data card, and discuss your thinking.	**1.** Silently read your card and think about what information you need to solve the problem. **2.** Ask your partner for the specific information that you need. **3.** Explain to your partner how you are using the information to solve the problem. **4.** When you have enough information, share the problem card with your partner, and solve the problem independently. **5.** Read the data card, and discuss your thinking.

Pause here so your teacher can review your work. You will be given a new set of cards to repeat the activity, trading roles with your partner.

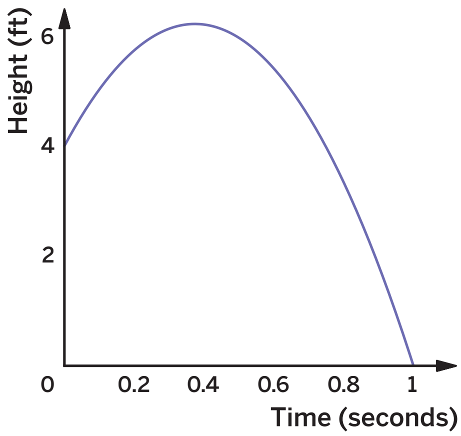
STOP

Name: ________ Date: ________ Period: ________

Summary

In today's lesson . . .

You interpreted key features of the graphs of quadratic functions representing the paths of objects that are launched in the air. As an example, the graph shows the height of a tennis ball that is launched into the air as a function of time.



In the graph, you can see some information you already know, and some new information:

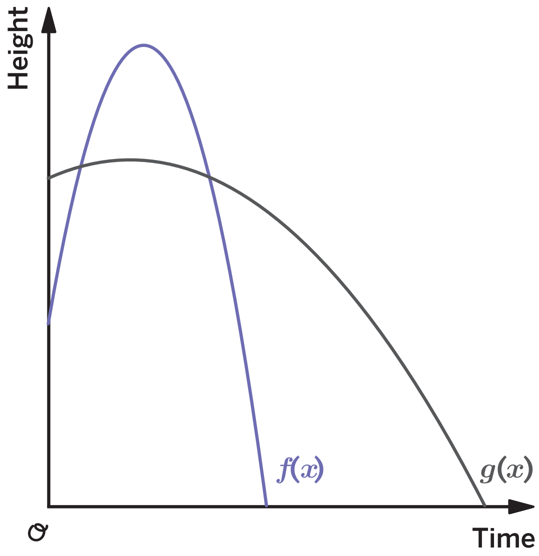
- The y-intercept represents the starting point of an object at time 0.
- The positive x-intercept represents where an object lands or hits the ground (or floor, water, etc.).
- The domain is restricted for this graph because only positive values of time are meaningful.
- The vertex is the maximum or minimum point of the graph. In this scenario, it represents the maximum height of the ball and the time at which the ball reaches its maximum height.

Reflect:

Practice

Name: .... Date: .... Period: ....

1. Refer to the graphs of functions f and g. Each represents the height of an object being launched into the air as a function of time.



a. Which object landed last? Explain your thinking.

b. Which object reached a higher point? Explain your thinking.

c. Which object was launched from a higher point? Explain your thinking.

2. The function $h(t) = -16(t - 1)(t + 0.5)$ models the height of a ball in feet, t seconds after it was thrown.

a. Determine the zeros of the function. Show or explain your thinking.

b. What do the zeros tell you about this scenario? Are both zeros meaningful?

c. From what height is the ball thrown? Explain your thinking.

Name: ____________ Date: ____________ Period: ____________

Practice

3. The height, in feet, of a thrown football is modeled by the function $f(t) = 6 + 30t - 16t^2$, where t represents the time, in seconds.

a What does the constant 6 mean in this scenario?

b What does the term $30t$ mean in this scenario?

c How does the squared variable term $-16t^2$ affect the value of the function f? What does this term reveal about the scenario?

4. Predict the x- and y-intercepts of the graph of the quadratic function defined by $f(x) = (x + 6)(x - 6)$. Explain how you made your predictions.

5. Consider the functions $f(x) = 13x + 6$ and $g(x) = 0.1 \cdot (1.4)^x$.

a Which function eventually grows faster, f or g? Explain your thinking.

b Use graphing technology to determine where in the first quadrant the graphs of f and g meet.

6. Expand the function $f(x) = -2(x + 4)^2 - 3$ into standard form. What is the function's y-intercept?

Vertex Form

Let's find out about the vertex form.

Warm-up Notice and Wonder

Study the functions in each set. What do you notice? What do you wonder?

Set 1	Set 2
$f(x) = x^2 + 4x$	$p(x) = -x^2 + 6x - 5$
$g(x) = x(x + 4)$	$q(x) = (5 - x)(x - 1)$
$h(x) = (x + 2)^2 - 4$	$r(x) = -1(x - 3)^2 + 4$

1. I notice . . .

2. I wonder . . .

Log in to Amplify Math to complete this lesson online.

Name: ____________ Date: ____________ Period: ____________

Activity 1 A Whole New Form

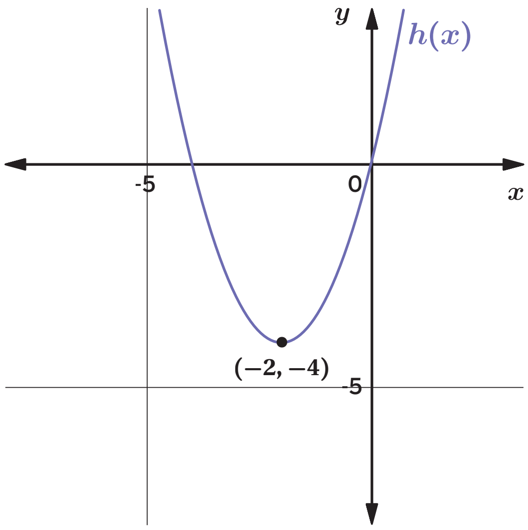
Here are two sets of quadratic functions you saw earlier. In each set, the functions are equivalent.

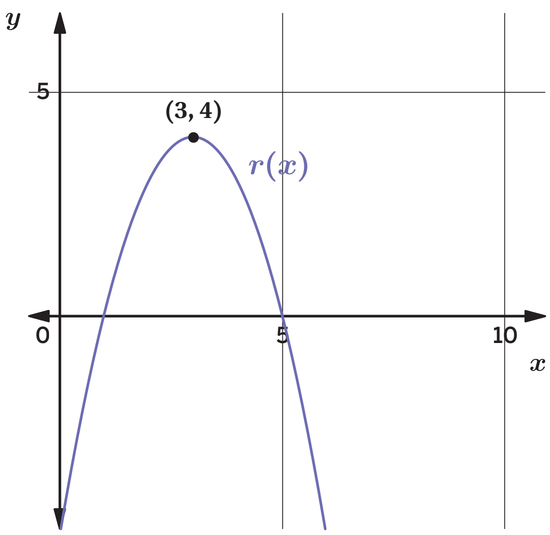
Set 1	Set 2
$f(x) = x^2 + 4x$ $g(x) = x(x + 4)$ $h(x) = (x + 2)^2 - 4$	$p(x) = -x^2 + 6x - 5$ $q(x) = (5 - x)(x - 1)$ $r(x) = -1(x - 3)^2 + 4$

1. The function $h(x)$ is written in ***vertex form***. Show that it is equivalent to $f(x)$.

2. Show that the functions $r(x)$ and $p(x)$ are equivalent.

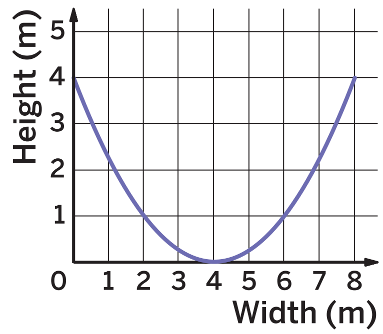
3. Refer to the graphs representing the quadratic functions $h(x)$ and $r(x)$. Why do you think the functions $h(x)$ and $r(x)$ are said to be written in vertex form?





Activity 2 Half-Pipe

The half-pipe ramp at a skateboard park is in the shape of a wide parabola that can be described in standard form by the equation $y = \frac{1}{4}x^2 - 2x + 4$, and in vertex form by $y = \frac{1}{4}(x - 4)^2$. The graph of this relationship is shown where y represents the height in meters above the ground and x represents the horizontal distance, in meters, from one edge of the half-pipe. The domain of the function has been restricted as shown.



1. Study the graph. What is the y-intercept and what does it tell you about the height of the ramp?

2. Which form tells you the y-intercept? Explain or show your thinking.

3. If the ramp begins at the y-intercept, what are the coordinates of the point representing the end of the ramp?

4. What are the coordinates of the vertex? Describe where it is located on the graph.

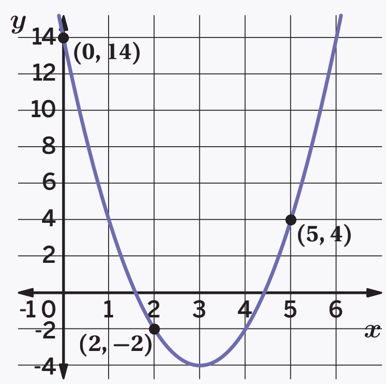
5. Use the x-coordinate of the vertex to determine the total width of the ramp. Explain or show your thinking.

6. In which form can you easily determine the x-coordinate of the vertex?

Are you ready for more?

1. What is the vertex of this graph?

2. Write an equation whose graph has the same vertex and adjust it, if needed, so that it can be represented by the graph shown.



STOP

Name: ______ Date: ______ Period: ______

Summary

In today's lesson . . .

You studied another form of a quadratic function, the ***vertex form***. You also saw the connections between the standard and vertex forms of quadratic functions. Both forms have constant terms; however, these two constant terms are not visible on the graph in the same way.

The standard form, $ax^2 + bx + c$, has the constant term c that tells you the y-intercept. The vertex form, $a(x - h)^2 + k$ has the constant term, k that tells you the y-coordinate of the vertex.

Changing these constant terms moves the graph up or down. In both the vertex and standard form, the squared variable term has a coefficient, which indicates whether the graph opens upward or downward, and whether the graph is wider or narrower.

Reflect:

Practice

Name: .... Date: .... Period: ....

1. Select *all* the quadratic expressions written in vertex form.

   A. $x^2 - 4$

   B. $(x + 3)^2$

   C. $x(x + 1)$

   D. $(x - 2)^2 + 1$

   E. $(x - 4)^2 + 6$

2. Consider the functions $m(x) = x(x + 6)$ and $p(x) = (x + 3)^2 - 9$.

   a. Show that $m(x)$ and $p(x)$ are equivalent.

   b. What is the vertex of the graph of $m(x)$? Explain your thinking.

   c. What are the x-intercepts of the graph of $p(x)$? Explain your thinking.

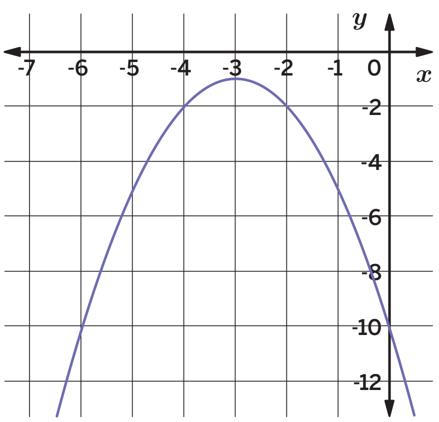
3. Which equation is represented by the graph?

   A. $y = (x - 1)^2 + 3$

   B. $y = (x - 3)^2 + 1$

   C. $y = -(x + 3)^2 - 1$

   D. $y = -(x - 3)^2 + 1$

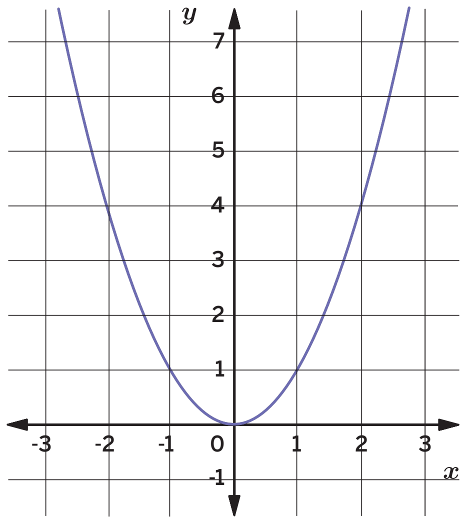


4. At 6:00 a.m., Lin began hiking. By noon, she had hiked 12 miles. By 4:00 p.m., Lin had hiked a total of 26 miles. During which time interval was Lin hiking faster? Explain your thinking.

Practice

Name: .... Date: .... Period: ....

5. Refer to the graph that represents $y = x^2$. Describe what happens to the graph when the original equation is modified as follows:



a $y = -x^2$

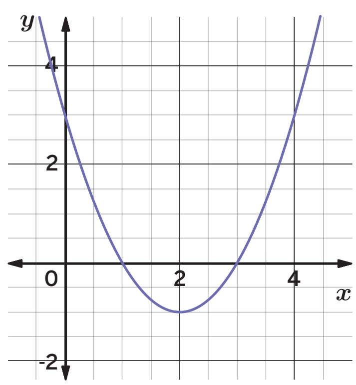
b $y = 3x^2$

c $y = x^2 + 6$

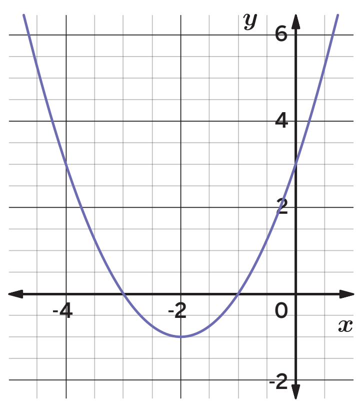
d Sketch the graph of the equation $y = -3x^2 + 6$ on the same coordinate plane as $y = x^2$.

6. Match each graph with the equation that it represents.

a



b



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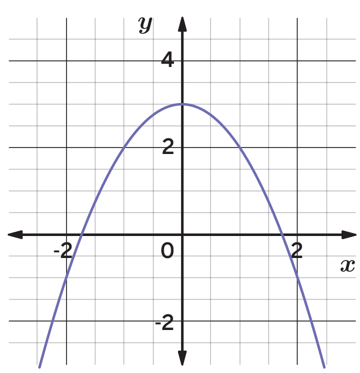

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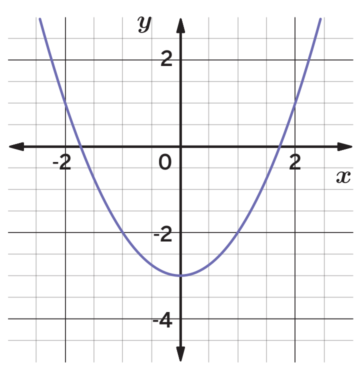


Equation

.... $y = -x^2 + 3$

.... $y = (x + 1)(x + 3)$

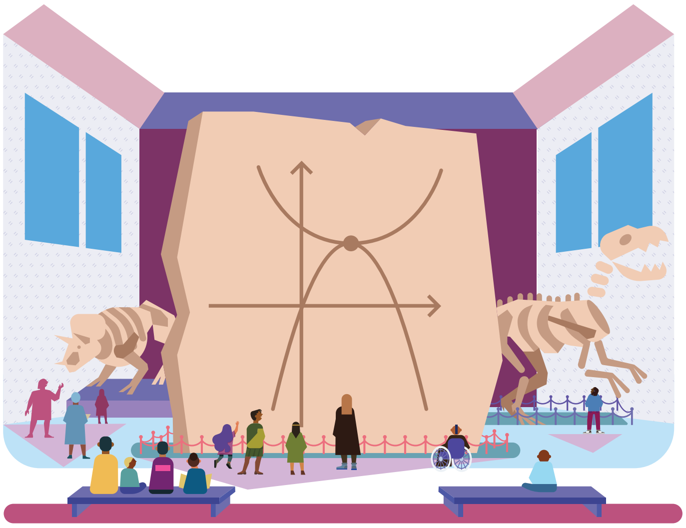
.... $y = x^2 - 3$

.... $y = (x - 1)(x - 3)$

Unit 5 | Lesson 20

Graphing With the Vertex Form

Let's graph functions using vertex form.



Warm-up Connecting Representations

Without using graphing technology, match each of the three functions in factored form with its equivalent function in vertex form.

Factored form

Function A	Function B	Function C
$f(x) = (x - 3)(x + 5)$	$g(x) = (x - 3)(x - 5)$	$h(x) = (x + 3)(x - 5)$

Vertex form

Function 1	Function 2	Function 3
$p(x) = (x - 1)^2 - 16$	$q(x) = (x - 4)^2 - 1$	$r(x) = (x + 1)^2 - 16$

1. Function A matches ____________ .
2. Function B matches ____________ .
3. Function C matches ____________ .

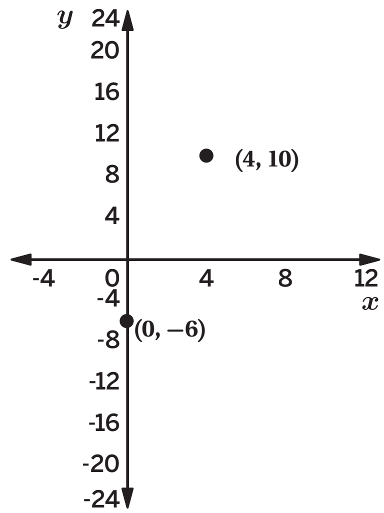
Log in to Amplify Math to complete this lesson online.

Name: ______ Date: ______ Period: ______

Activity 1 Sharing a Vertex

Consider the quadratic functions $p(x) = -(x-4)^2 + 10$ and $q(x) = \frac{1}{2}(x-4)^2 + 10$.

1. The graph of $p(x)$ passes through $(0, -6)$ and $(4, 10)$, as shown on the coordinate plane. Find the coordinates of another point on the graph of $p(x)$. Explain or show your thinking. Use the points to sketch and label the graph.



2. On the same coordinate plane, identify the vertex and two other points that are on the graph of $q(x)$. Explain or show your thinking. Sketch and label the graph of $q(x)$.

3. Priya says, "Once I know the coordinates of the vertex, I can determine, without graphing, whether the vertex is the maximum or the minimum of the function $p(x)$. I can just compare the coordinates of the vertex with coordinates of a point on either side of it."

   Complete the table and then explain how Priya might have reasoned about whether the vertex is the minimum or maximum.

x	3	4	5
$p(x)$		10	

Activity 2 Card Sort: Matching Functions With Graphs

You will be given a set of cards. Each card contains a graph or a quadratic function. Take turns matching each graph to a function.

- For each pair of cards that you match, explain to your partner how you know they belong together.
- For each pair of cards that your partner matches, listen carefully to their explanation. If you disagree, discuss your thinking and work to reach an agreement.
- Once all the cards are matched, record the function, sketch the corresponding graph, and write a brief note about how you knew they were a match.

This function . . .	matches this graph . . .	because . . .

Name: ______ Date: ______ Period: ______

Activity 2 Card Sort: Matching Functions With Graphs (continued)

This function . . .	matches this graph . . .	because . . .

Activity 3 Match My Parabola

For each Challenge in the digital activity, write the equation for the requested parabola.

Challenge 1:

Parabola	Equation
Red	
Blue	
Green	
Orange	
Purple	

Challenge 2:

Challenge 3:

Parabola	Equation
Red	
Blue	
Green	

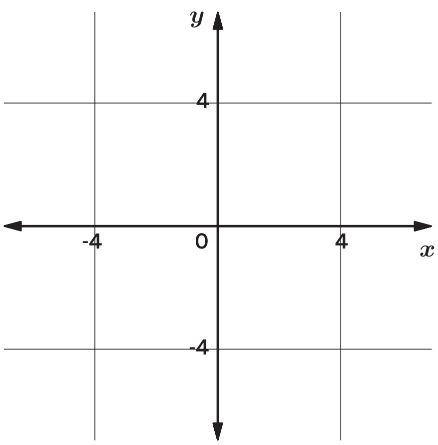
Challenge 4:

Parabola	Equation
Red	
Blue	
Green	
Orange	
Purple	

Challenge 5:

Challenge 6:

Sketch the graph of:



STOP

Name: ______ Date: ______ Period: ______

Summary

In today's lesson . . .

You saw that vertex form is helpful for determining the vertex of the graph of a quadratic function. An equation of the form $y = a(x - h)^2 + k$ has its vertex at (h, k).

When the coefficient a is positive, the graph opens upward and the vertex represents the minimum value of the function. When a is negative, the graph opens downward and the vertex represents a maximum value.

To determine the y-intercept of the graph of a quadratic function (or any function), evaluate the function at $x = 0$. You can determine other points on the graph with the same y-coordinate using the graph's axis of symmetry.

Reflect:

Practice

Name: .... Date: .... Period: ....

1. Which equation can be represented by a graph with a vertex at $(1, 3)$?

   A. $y = (x - 1)^2 + 3$

   B. $y = (x + 1)^2 + 3$

   C. $y = (x - 3)^2 + 1$

   D. $y = (x + 3)^2 + 1$

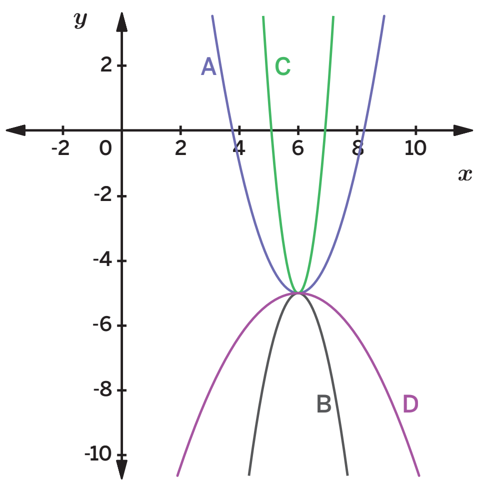
2. Match each graph with the equation that represents it.

   .... $y = -2(x - 6)^2 - 5$

   .... $y = (x - 6)^2 - 5$

   .... $y = 6(x - 6)^2 - 5$

   .... $y = -\frac{1}{3}(x - 6)^2 - 5$



3. Andre thinks the vertex of the graph of the equation $y = (x + 2)^2 - 3$ is located at $(2, -3)$. Lin thinks the vertex is located at $(-2, 3)$. Do you agree with either of them?

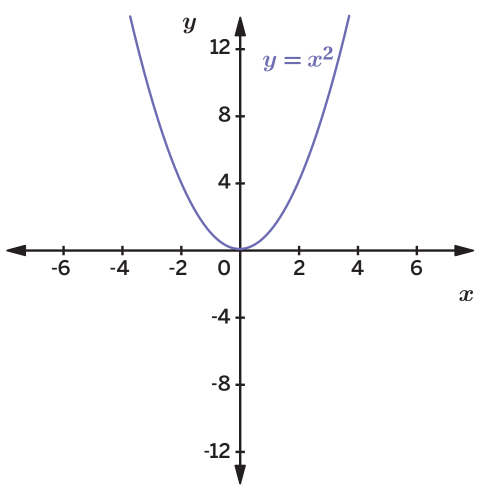
4. Refer to the graph of $y = x^2$.

   a. Describe what would happen to the graph if the original equation were changed to:

   $y = \frac{1}{2}x^2$:

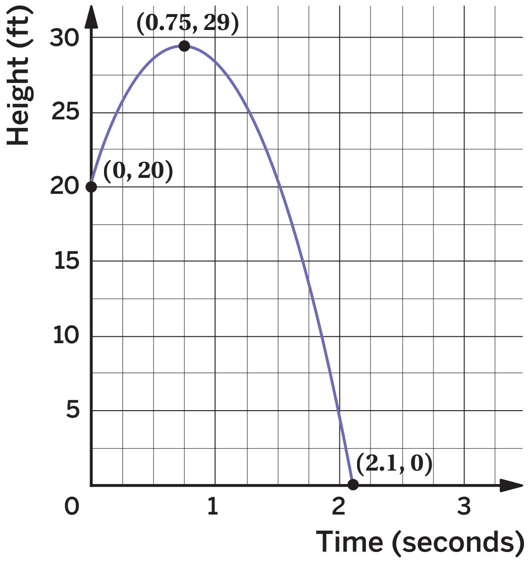
   $y = x^2 - 8$:

   b. Graph the equation $y = \frac{1}{2}x^2 - 8$ on the same coordinate plane as $y = x^2$.



Name: ______________________ Date: __________ Period: __________

5. Clare throws a rock into the lake. The graph shows the rock's height above the water, in feet, as a function of time in seconds.



Select *all* the statements that describe this situation.

A. The vertex of the graph is $(0.75, 29)$.

B. The y-intercept of the graph is $(2.1, 0)$.

C. Clare just dropped the rock into the lake.

D. The maximum height of the rock is about 20 ft.

E. The rock hits the surface of the water at 2.1 seconds.

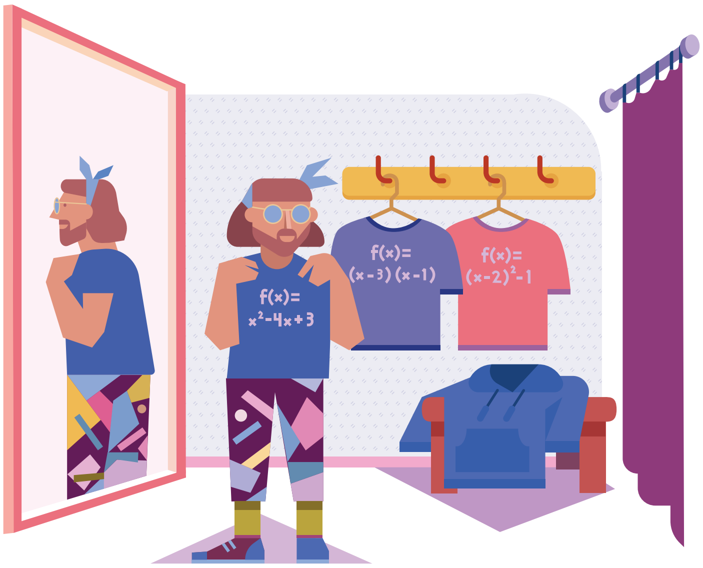
F. Clare tossed the rock up into the air from a point 20 ft above the water.

6. Consider the function $v(x) = \frac{1}{2}(x + 5)^2 - 7$. Without graphing, determine whether the vertex of the graph representing $v(x)$ shows the minimum or maximum value of the function. Explain your thinking.

Unit 5 | Lesson 21

Changing Parameters and Choosing a Form

Let's change the parameters of quadratic functions and examine the usefulness of different forms.



Warm-up Which Form to Use?

Different forms of quadratics can be used to define the same function. Here are three ways to define a function f.

Standard form: $f(x) = x^2 - 4x + 3$

Factored form: $f(x) = (x - 3)(x - 1)$

Vertex form: $f(x) = (x - 2)^2 - 1$

Which form would you use if you wanted to determine the following features of the graph of f? Explain your thinking.

1. The x-intercepts.

2. The vertex.

3. The y-intercept.

Name: ____________ Date: ________ Period: ________

Activity 1 Playing With Parameters

1. Using graphing technology, graph the equation $y = x^2$. Then add and subtract different numbers to x before it is squared. For example: $y = (x + 4)^2$ or $y = (x - 3)^2$. Observe how the graph changes, and record your observations.

2. Graph $y = x^2$. Experiment with each of the following changes to the equation to see how they affect the graph.

   a Add or subtract different constant terms to x^2.
   (For example: $y = x^2 + 5$ or $y = x^2 - 9$).
   Record your observations.

   b Multiply x^2 by different positive and negative coefficients.
   (For example $y = 3x^2$ or $y = -2x^2$).
   Record your observations.

Activity 1 Playing With Parameters (continued)

3. Without graphing, predict the coordinates of the vertex of the graphs of these functions, whether the graph opens upward or downward, and whether the graph is wider or narrower compared to the function $y = x^2$.

Function	Coordinates of the vertex	Opens upward or downward?	Wider or narrower?
$y = 0.5(x + 10)^2$			
$y = 5(x - 4)^2 + 8$			
$y = -7(x - 4)^2 + 8$			
$y = 6x^2 - 7$			
$y = \frac{1}{2}(x + 3)^2 - 5$			
$y = -3(x + 100)^2 + 50$			

4. Use graphing technology to check your predictions. If any of your predictions are incorrect, revise them.

5. In the equation $y = a(x + m)^2 + n$, how do the values of a, m, and n affect the shape of the graph?

a a:

b m:

c n:

Name: ______________________ Date: __________ Period: __________

Activity 2 Shifting the Graph

1. How would you change the equation $y = x^2$ so that the vertex of its graph were located at the following coordinates and the graph opens as described?

   a. $(0, 11)$, opens upward

   b. $(7, 11)$, opens upward

   c. $(7, -3)$, opens downward

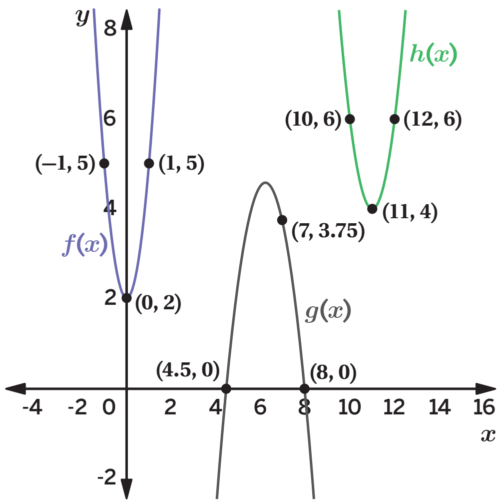
2. Use graphing technology to verify your predictions. Adjust your equations if necessary.

3. Kiran graphed the equation $y = x^2 + 1$ and noticed that the vertex is located at $(0, 1)$. He changed the equation to $y = (x - 3)^2 + 1$ and saw that the graph shifted 3 units to the right and the vertex is now at $(3, 1)$. Next, he graphed the equation $y = x^2 + 2x + 1$, and observed that the vertex is located at $(-1, 0)$. Kiran thought, "If I change x^2 to $(x - 5)^2$ in the equation $y = x^2 + 2x + 1$, the graph will move 5 units to the right and the vertex will be located at $(4, 0)$." Do you agree with Kiran? Explain or show your thinking.

Stronger and Clearer: Share your responses to Problem 3 with another pair of students. Do your responses talk about the structure of the equations? Use the feedback you receive to revise your response.

Activity 3 Building a Function

Refer to the graphs of the three different quadratic functions, $f(x)$, $g(x)$, and $h(x)$. Three points are indicated on each graph.



1. What information and key features do you observe for each function?

   a. $f(x)$

   b. $g(x)$

   c. $h(x)$

2. Of the three forms of quadratic functions covered in this unit (vertex form, factored form, and standard form), which would you use to write an expression for each function? Explain your thinking.

   a. $f(x)$

   b. $g(x)$

   c. $h(x)$

Reflect: How can you organize the information from this activity in a way that can help you recall and apply it later?

STOP

Name: ______ Date: ______ Period: ______

Summary

In today's lesson . . .

You observed how changing the parameters in a quadratic function changes its graph. For example, if you compare the graphs of $f(x) = 2x^2$ and $g(x) = -0.5(x - 2)^2 + 3$, you see that $g(x)$ is wider compared to $f(x)$, because the coefficient of the squared term in $g(x)$ is less than the coefficient of the squared term in $f(x)$. Also, $g(x)$ is shifted 2 units to the right and 3 units up, has a vertex of $(2, 3)$, and opens downward because the coefficient of the squared term is negative.

When creating a quadratic function to represent a given graph, some quadratic forms may be more useful than others, depending on the information provided in the graph.

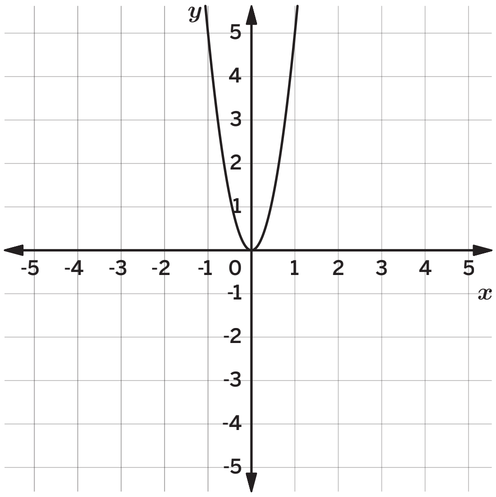
For example, if you can identify the vertex from the graph, you may want to write the function in vertex form. However if you can identify the x-intercepts from the graph, you may want to write the function in factored form with an unknown coefficient.

Reflect:

Practice

Name: .... Date: .... Period: ....

1. The graph of $y = 5x^2$ is given. Graph $y = \frac{1}{2}(x - 3)^2$ on the same coordinate plane.

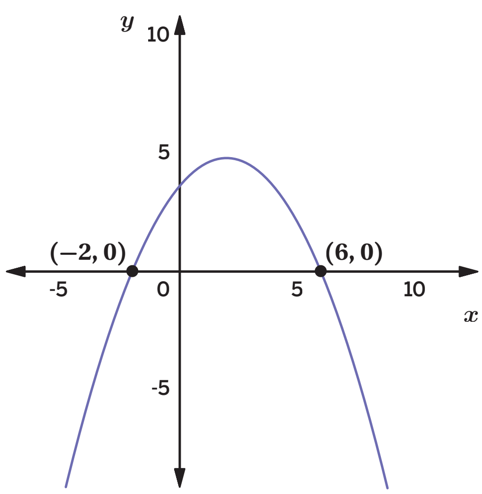


2. For the equation $y = 3x^2 + 1$, describe what would happen to the graph if the equation was changed to:

a $y = 2x^2 + 1$

b $y = 6x^2 + 8$

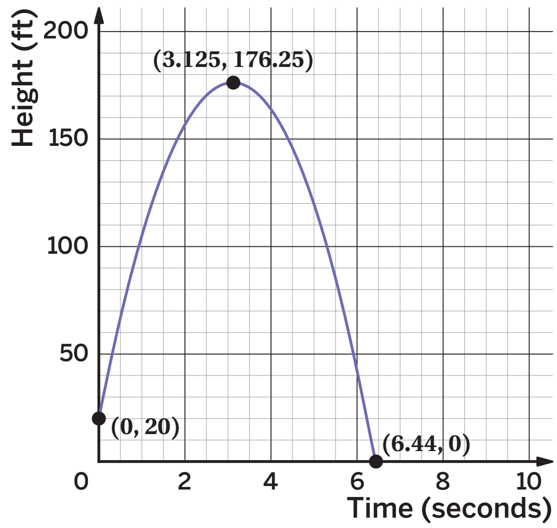
3. The graph of $g(x)$ is given. Which form of a quadratic function would you use to write this function? Explain your thinking.



Name: ________________ Date: ________ Period: ________

Practice

4. Mai stands at the edge of a cliff and launches a model rocket into the air. The graph shows the rocket's height above the ground, in feet, as a function of time in seconds. Select *all* the statements that describe this situation.



A. The rocket reaches its maximum height at 3.125 seconds.

B. The cliff is 20 ft high.

C. The y-intercept of the graph is 6.44.

D. The rocket hits the ground after about 3.125 seconds.

E. The rocket was in the air for about 6.44 seconds.

5. The expression $2000 \cdot (1.15)^5$ represents the balance, in dollars, in a savings account after 5 years.

a. What is the rate of interest paid on the account?

b. How much money was invested?

c. How much money is in the account now?

6. The function $g(x)$ is created by graphing the function $f(x) = (x - 4)^2 + 1$ and shifting it to the right 2 units, down 3 units, and reflecting it across its vertex so that it opens downward. What is the function $g(x)$?

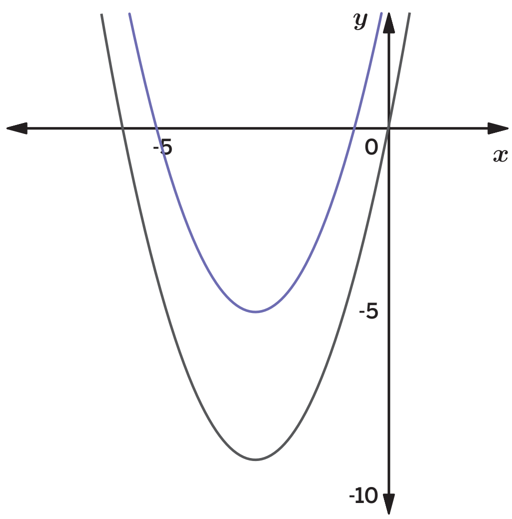
Unit 5 | Lesson 22

Changing the Vertex

Let's write new quadratic functions in vertex form to make specific graphs.

Warm-up Two Functions

Refer to the graphs representing the functions $f(x) = x(x + 6)$ and $g(x) = x(x + 6) + 4$.



1. Which graph represents each function? Explain or show your thinking.

2. Where does the graph of $f(x)$ meet the x-axis? Explain or show your thinking.

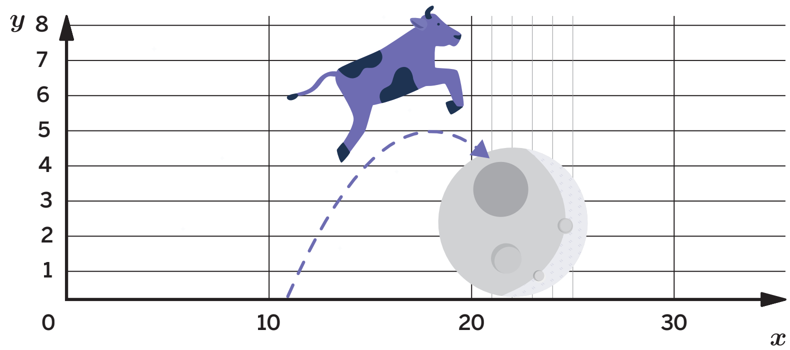
Log in to Amplify Math to complete this lesson online.

Name: ______ Date: ______ Period: ______

Activity 1 The Cow Jumped Over the Moon

Mai is learning how to write code for computer animations. She is animating her sister's favorite nursery rhyme and uses the equation, $y = -0.1(x - h)^2 + k$ to model a cow jumping over the moon, where y represents the height of the cow and x is the horizontal distance traveled. In her animation, one diameter of the Moon has endpoints at the coordinates $(22, 0)$ and $(22, 4.5)$.

The dashed curve on the graph models Mai's first attempt to animate the cow jumping over the full Moon.



1. What are some possible values of h and k in Mai's original equation?

2. Select values for h and k that will guarantee the cow stays on the screen but also jumps over the Moon. Explain or show your thinking.

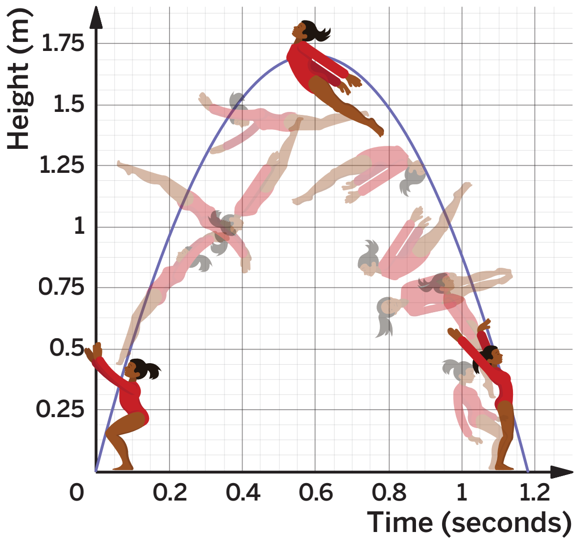
Activity 2 Triple-Double

Gymnast Simone Biles made history as the first woman to land a triple-double tumble (now known as "The Biles") during her floor routine exercise at the 2019 U.S. Gymnastics Championships.

Leonard Zhukovsky/Shutterstock.com

The change in height of Biles' center of mass, in meters, can be modeled by the equation $y = 5.78x - 4.9x^2$. A graph of this equation is shown.

1. Using the graph and the equation, what is the y-intercept? Explain its meaning in the context of Biles' jump.



2. Study the graph. Approximately when did Simone reach her peak during the jump? Approximately how high was her jump?

Name: ______ Date: ______ Period: ______

Activity 2 Triple-Double (continued)

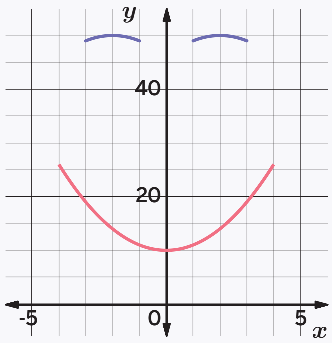
3. Approximately how long was Biles in the air? Explain your thinking.

4. Starting with the equation $y = -4.9(x - h)^2 + k$, write an equation in vertex form that models her height in meters as a function of time in seconds.

5. If Biles increases her launch speed to 7 m/second, she will be in the air for 1.43 seconds. Using the equation $y = -4.9(x - 0.715)^2 + 2.5$, what would be her maximum height in the air and when would she reach it?

Are you ready for more?

Do you see 2 "eyes" and a smiling "mouth" on the graph? The 3 arcs on the graph all represent quadratic functions that were initially defined by $y = x^2$, but whose equations were later modified.



1. Write equations to represent each curve in the smiley face.

2. What domain is used for each function to create this graph?

STOP

Summary

In today's lesson . . .

You used vertex form, $y = a\,(x - h)^2 + k$, to write functions to represent certain graphs. The graph that represents $f(x) = x^2$ has its vertex at $(0, 0)$.

The graphs of $f(x) = x^2$, $f(x) = x^2 + k$ and $f(x) = (x + h)^2$ all have the same shape, but their locations are different. Adding a constant number k to x^2 raises the graph by k units, so the vertex of that graph is now at $(0, k)$. Replacing x^2 with $(x + h)^2$ shifts the graph h units to the left, so the vertex is now at $(-h, 0)$.

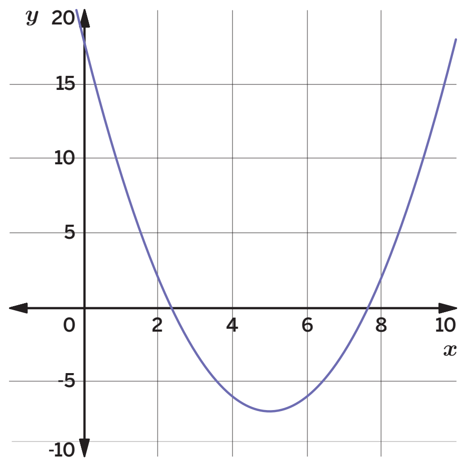
You can also shift a graph horizontally and vertically at the same time. The graph that represents $f(x) = (x - h)^2 + k$ will look the same as the graph for $f(x) = x^2$, but it will be shifted k units up and h units to the right. Its vertex is located at (h, k).

When the x^2 term is multiplied by a negative number, the graph is flipped, or reflected, across a horizontal line, so that it opens downward.

Reflect:

Name: .... Date: .... Period: ....

1. Refer to the graph of the quadratic function $f(x)$. Andre says that $f(x) = (x - 5)^2 + 7$ and Noah says that $f(x) = (x + 5)^2 - 7$. Do you agree with either of them? Explain your thinking.

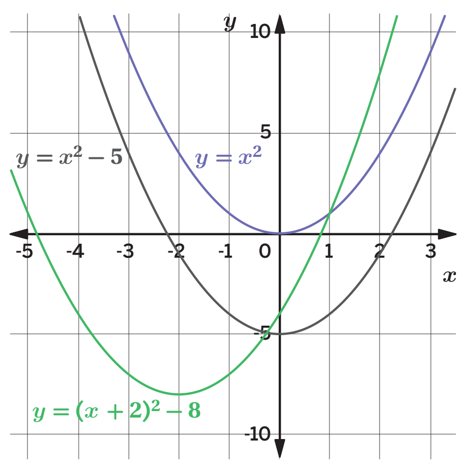


2. Refer to the graphs of the equations $y = x^2$, $y = x^2 - 5$, and $y = (x + 2)^2 - 8$.

   a. How do the three graphs compare?

   b. How does the term -5 in the equation $y = x^2 - 5$ affect the graph of $y = x^2$?

   c. How do the terms $+2$ and -8 in the equation $y = (x + 2)^2 - 8$ affect the graph of $y = x^2$?



3. The height, in feet, of a soccer ball is modeled by the function $g(t) = 2 + 50t - 16t^2$, where t represents the time, in seconds, after the ball was kicked.

   a. How far above the ground was the ball when it was kicked?

   b. What was the initial upward velocity of the ball?

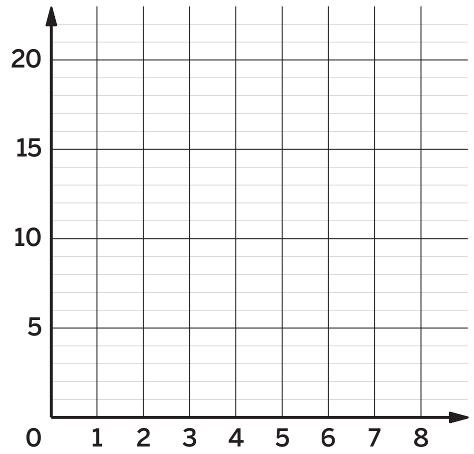
   c. Why is the coefficient of the squared variable term negative?

Practice

Name: ______ Date: ______ Period: ______

4. Kiran bought one smoothie every day for a week. Smoothies cost $3 each. The total amount of money he has spent, in dollars, is a function of the number of days of buying smoothies.

a Sketch a graph of this function. Be sure to label the axes.

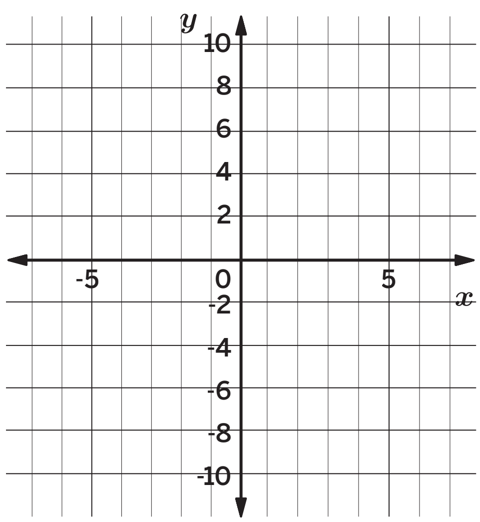


b Describe the domain and range of this function.

5. Consider the function $f(x) = -(x - 3)^2 + 6$.

a Identify the vertex, y-intercept, and one other point on the graph of $f(x)$.

b Sketch the graph of $f(x)$.



Name: .... Date: .... Period: ....

Unit 5 | Lesson 23 – Capstone

Monster Ball

Let's use our knowledge of quadratic functions and their graphs to play monster ball!

Warm-up Have You Ever Played Monster Ball?

A giant "monster ball" is placed in the center of a rectangular court and there are two teams. Each team is trying to move the ball so it passes outside of the rectangle on the opposing team's half of the court. Sounds easy, right?

The twist is you cannot *touch* the monster ball. To move it, you must throw smaller balls at it. Once it passes outside the rectangle on your opponent's half of the court, your team gets a point, and the ball is placed back in the center of the court for the next round.

Players . . .

- **Can go anywhere to retrieve a ball, even inside the rectangle.**
- **Can only throw balls from outside the rectangle.**
- **Cannot touch the monster ball.**
- **Cannot block any thrown balls from the other team.**

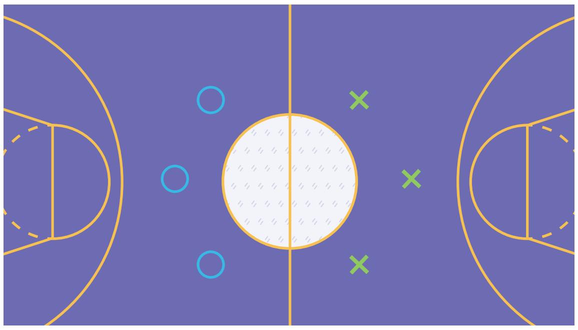
1. What should you and your team consider when trying to move the monster ball?

2. What questions do you have about the game and the throws?

Log in to Amplify Math to complete this lesson online.

Activity 1 Know the Court

The perimeter of a typical rectangular monster ball court is 268 ft. Assume your team consists of 10 players.



1. Consider the rectangle whose perimeter is 268 ft that has the maximum possible area.

   a What is this maximum possible area? What are the dimensions of this rectangle?

   b If each member on your team is responsible for collecting balls from different regions of equal area inside the rectangle, what is that area?

2. For a court with a length of x, write a function that represents the area inside the court that each of the 10 players is responsible for. Explain how you determined the function.

3. For a team consisting of b players, write a function that could be used to calculate the size of the area inside the court each player is responsible for.

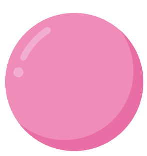
4. A typical high school basketball court has a length of 84 ft. If the game were played on a high school basketball court, draw a sketch of how your team would divide up the court so that each player was responsible for an equal area.

Name: ______________________ Date: __________ Period: __________

Activity 2 The Perfect Throw

You and your teammates line up along the perimeter of the rectangle, ready to throw. There are four different balls you can throw at the monster ball: a tennis ball, soccer ball, kickball, and basketball. Each of these balls is thrown at a specific initial vertical and horizontal speed, as shown in the table.

After t seconds, the height $f(t)$ of a ball that is thrown from an initial height h and with an initial vertical speed v is given by the function $f(t) = -16t^2 + vt + h$.

Ball	Weight (g)	Initial vertical speed (ft/second)	Initial horizontal speed (ft/second)
Tennis ball 	58	9.6	54.2
Soccer ball 	450	6.1	34.5
Kickball 	548	5.2	29.6
Basketball 	623	3.5	19.7

Activity 2 The Perfect Throw (continued)

1. Your teammate is throwing a tennis ball from a height of 3 ft.

a Write a function to model the height of the tennis ball as a function of time.

b You can determine the horizontal distance a ball travels by multiplying its time spent in the air by its horizontal speed. How far will the tennis ball have horizontally traveled when it hits the ground? Explain your thinking.

2. Another teammate throws her ball from a height of 6 ft. She wants her throw to directly hit the front of the monster ball, which is 1 ft off the ground. How far from the monster ball should she throw each of the following balls? Explain your thinking.

a Basketball:

b Kickball:

c Soccer ball:

Name: .... Date: .... Period: ....

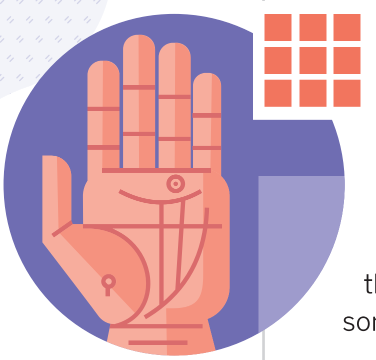
Activity 3 Let's Play!

The planning is over, and now it is your turn to play monster ball. Use both your strength and your mathematical brilliance to help your team win the game. On this page, record any observations about your team's strategy and how you threw the balls for maximum effect.

Good luck out there!

STOP

Unit Summary

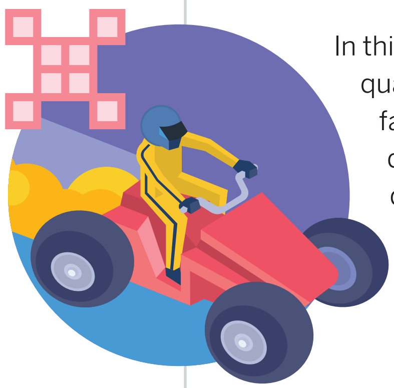


We can go through most of our lives without thinking too much about gravity. Yet it's with us everywhere, affecting nearly everything we see. Whether it is sweet jump shots or shooting a clown out of a cannon, gravity influences the way objects move through space.

Perhaps that object is a tennis ball lobbed over a net. Or it is the human form, like Simone Biles during her record-shattering somersault that forever changed gymnastics.

Instinctively, we naturally understand gravity. It is how we predict where a falling object will land to catch it. We recognize the particular curve an object takes through the air — what is called a "parabola."

This parabola is a surefire sign of a "quadratic" relationship, where one expression is squared to get another.



In this unit you saw different ways to express quadratics: in standard form, in vertex form, and in factored form. And each of these shows us something different about the quadratic: whether it opens up or down, its vertex, or where it crosses the axes.

So while we may understand gravity's effects through intuition, algebra provides the vocabulary to help us articulate this phenomena more precisely and expressively. By understanding the math behind this "new kind of change," we have a new framework for appreciating the grace and beauty of how objects move.

See you in Unit 6.

Name: .... Date: .... Period: ....

Practice

1. Kiran plays on his high school baseball team and is practicing his throw before the game. He releases the ball from a height of 5.5 ft with an initial vertical speed of 20 feet per second.

   a. Write a function to model the height of Kiran's throw as a function of time, in seconds.

   b. What does each term represent in the function?

2. Match each quadratic expression that is written in factored form with an equivalent expression that is written in standard form.

a	$(x+2)(x+6)$	.... $x^2 + 12x + 32$
b	$(2x+8)(x+2)$	.... $2x^2 + 16x + 24$
c	$(x+8)(x+4)$	.... $2x^2 + 12x + 16$
d	$2(x+2)(x+6)$	.... $x^2 + 8x + 12$

3. Which quadratic expression is written in standard form?

   A. $(x+10)x$

   B. $(x-4)^2 + 5$

   C. $-4x^2 - 10x + 19$

   D. $3x^2 + 5(x+1)$

4. Select *all* equations whose graphs have a y-intercept with a positive y-coordinate.

   A. $y = x^2 + 3x$

   B. $y = (x-4)^2$

   C. $y = (2x+1)(x+6)$

   D. $y = 5x^2 - 10x - 19$

Practice

Name: ________ Date: ________ Period: ________

5. Consider the two quadratic functions:

$$f(x) = (x+3)(x+1) \qquad g(x) = (x+2)^2 - 1$$

a. Show that the functions are equivalent.

b. What are the x-intercepts of the graph of g? Explain your thinking.

c. What is the vertex of the graph of f? Explain your thinking.

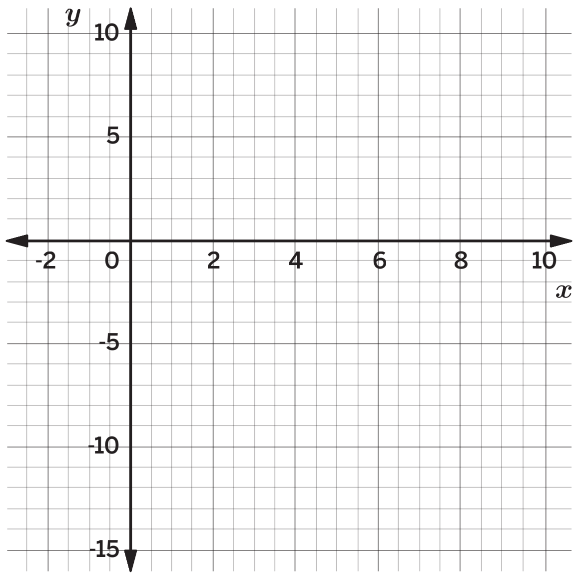
6. Consider the function $h(x) = -(x-4)^2 + 3$.

a. Where is the vertex of the graph located?

b. What is the y-intercept?

c. Does the graph open upward or downward? Explain your thinking.

d. Sketch a graph that represents the function.



My Notes:

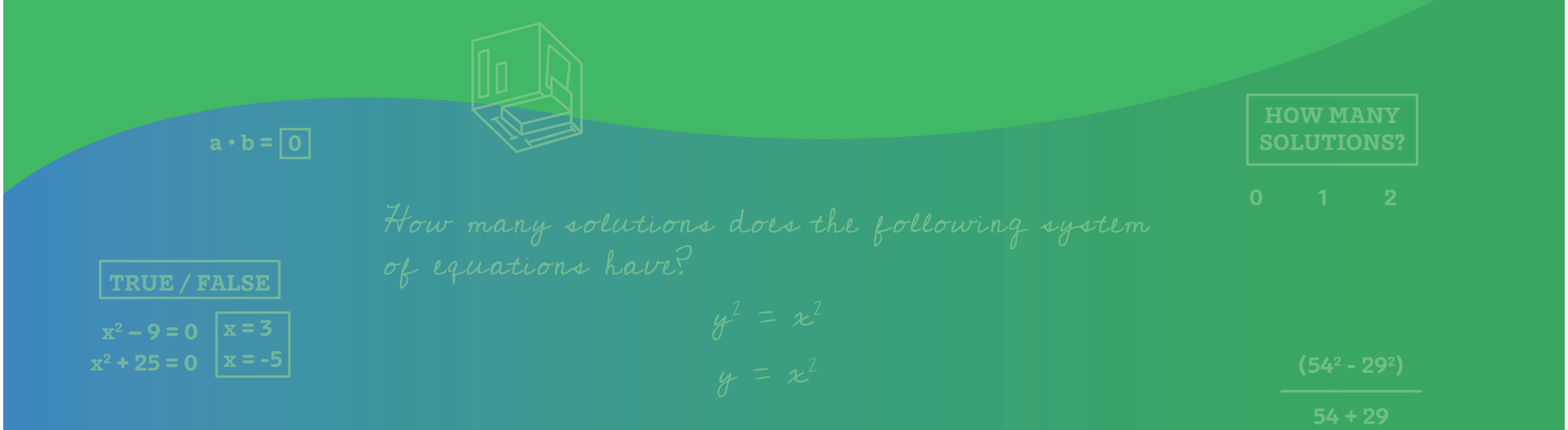
UNIT 6

Quadratic Equations

In this unit, you will explore how people have learned to solve quadratic equations throughout history. You will write, solve, and explore strategies for solving quadratic equations.

Essential Questions

- How does solving quadratic equations compare to solving linear equations?
- How does the structure of a quadratic equation determine efficient strategies for solving it algebraically?
- How are quadratic equations used to solve real-world problems?
- *(By the way, in which year was a new way to solve quadratic equations discovered: 628 CE or 2019 CE?)*

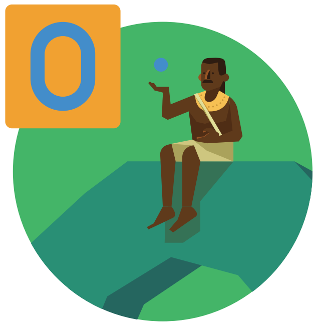


SUB-UNIT

1 Connecting Quadratic Functions to Their Equations

Narrative: The story of solving quadratic equations began with the need to calculate area.

You'll learn . . .
- about the meaning of a solution to a quadratic equation.



SUB-UNIT

2 Factoring Quadratic Expressions and Equations

Narrative: Discover what happens when you set a quadratic expression equal to zero.

You'll learn . . .
- how factoring can help you solve some quadratic equations.

SUB-UNIT

3 Completing the Square

Narrative: Turn to geometry and your old friend, the square, to solve quadratic equations.

You'll learn . . .
- how geometry can help solve quadratic equations.

SUB-UNIT

4 Roots and Irrationals

Narrative: Explore how the world beyond rational numbers relates to quadratic equations.

You'll learn . . .
- about irrational solutions to quadratic equations.

SUB-UNIT

5 The Quadratic Formula

Narrative: Al-khwarizmi documented a way to solve *any* quadratic equation.

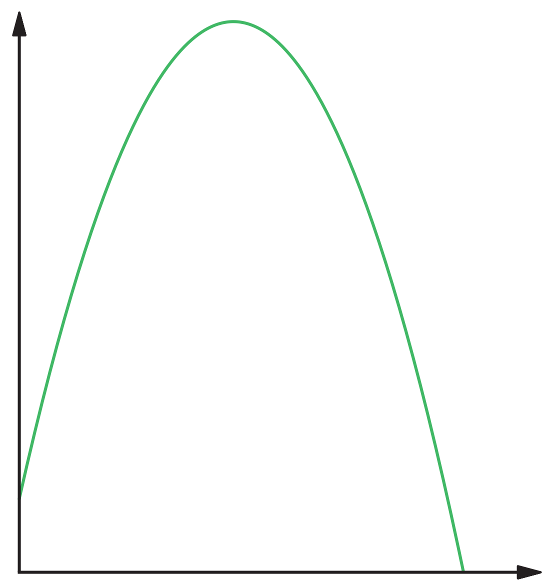
You'll learn . . .
- yet another way to solve quadratic equations.

Determining Unknown Inputs

Let's frame some pictures.

Warm-up Author Your Own Story

A referee, an athlete, and a ball are on a field. The graph shows the function $f(x)$, which models something that happened during practice. Write a story about what may have happened at practice, using the referee or the athlete as your main character. Be sure to label the axes to match your scenario.



Co-craft Questions:
Share your stories with a partner. Work together to write 2 – 3 mathematical questions you could ask about each of your stories.

My story . . .

Name: ______ Date: ______ Period: ______

Activity 1 Picture Framing

You will be given a picture measuring 7 in. by 4 in., framing material measuring 4 in. by 2.5 in., and a pair of scissors. Cut your framing material to make a frame for the picture, based on the following criteria:

- All of the framing material should be used, with no leftover pieces.
- The framing material should not overlap.
- The resulting frame should have the same thickness all the way around.

You will receive four copies of the framing material in case you need to refine your work.

Are you ready for more?

Han says, "The perimeter of the picture is 22 in. If I cut the framing material into nine pieces, each piece measuring 2.5 in. by $\frac{4}{9}$ in., I will be able to form a frame around the picture because these pieces will form a perimeter of 22.5 in."

Do you agree with Han? Explain your thinking.

Activity 2 Representing the Framing Problem

The diagram shows a picture with a frame that is the same thickness all the way around. The picture measures 7 in. by 4 in. The frame is created using 10 in^2 of framing material.

This diagram may not be drawn to scale.

CEPTAP/Shutterstock.com

Consider the picture and its surrounding frame as a single rectangle.

1. Write an equation to represent the relationship between the dimensions of this rectangle and its total area, where x represents the thickness of the frame.

2. What does a solution to your equation represent in this situation?

STOP

Unit 6 Quadratic Equations

The Evolution of Solving Quadratic Equations

There is a lot we take for granted. Even something as ordinary as the border of a picture frame can require careful thought and planning. And lurking inside what might at first appear to be a straightforward problem about area, we find our old pal, a quadratic.

In the previous unit, you learned that quadratics can take on different forms and represent different kinds of relationships. They can represent how objects fall through the air, how dancers twirl their bodies, or even how businesses generate profit. But humankind's pursuit of quadratics has its roots, if you will pardon the pun, in *area*.

The study of quadratics is an old story — thousands of years old — spanning civilizations across three continents. It had day-to-day implications on how land was divided and how buildings were planned out. In this unit, we will look at this story more closely. We will see how these societies tackled these problems, often using mathematics quite different from what we are used to today.

We will look at key moments across this wide swath of history, where problems of area inspired flashes of genius and innovation, giving birth to algebra as we know it.

Welcome to Unit 6.

Practice

Name: .... Date: .... Period: ....

1. Mai throws a paper airplane from her treehouse. The height the paper airplane reaches, in feet, is a function of time, in seconds. It can be modeled by the equation $h(t) = 25 + 2.5t - 0.5t^2$.

a Evaluate $h(0)$ and explain what this value means within the context of the problem.

b What would a solution to $h(t) = 0$ mean in this context?

c What does the equation $h(9) = 7$ mean in this context?

d What happens to the paper airplane 2.5 seconds after Mai throws it, if each of these statements is true?

$h(2) = 28$ $\qquad$ $h(2.5) = 28.125$ $\qquad$ $h(3) = 28$

2. A square picture has a frame that measures 3 in. in thickness all the way around. The total side length of the picture and its frame, together, is x in. Which expression represents the area, in square inches, of the square picture *without* the frame? Draw a diagram to help with your thinking, if needed.

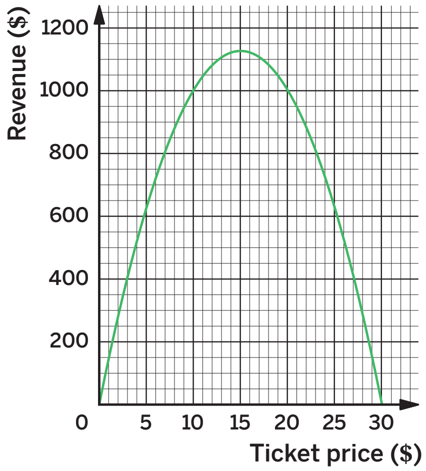
A. $(x - 6)(x - 6)$

B. $(x + 6)(x + 6)$

C. $(2x - 3)(2x - 3)$

D. $(2x + 3)(2x + 3)$

3. The revenue R from a youth league baseball game depends on the price per ticket x. The graph shown represents the revenue function $R(x)$. Select *all* statements that are true.



A. $R(600)$ is a little less than 5.

B. $R(5)$ is a little more than 600.

C. The maximum possible ticket price is $15.

D. The maximum possible revenue is about $1,125.

E. If the price of each ticket is $10, the predicted revenue is $1,000.

F. If the price of each ticket is $20, the predicted revenue is $1,000.

Name: ______ Date: ______ Period: ______

Practice

4. A random sample of people were asked to taste test two different types of frozen yogurt and give a taste score of either "low" or "high." The two types of frozen yogurt have similar recipes, only differing in the percentage of natural sweetener. What values could be used to complete the table so that it suggests there is an association between taste score and percentage of sweetener?

	12% sweetener	15% sweetener
Low taste score	239	
High taste score	126	

5. An American traveler, who is heading to Europe, is exchanging some U.S. dollars for European euros. At the time of his travel, 1 dollar can be exchanged for 0.91 euros.

a. How many euros would the American traveler receive if he exchanged \$100?

b. How much would he receive if he exchanged \$500?

c. Write a function that gives the number of euros as a function of the number of dollars d being exchanged.

d. Upon returning to America, the traveler has 42 euros to exchange back into U.S. dollars. How many dollars will he receive if the exchange rate is still the same?

e. Write a function that gives the number of dollars as a function of the number of euros e being exchanged.

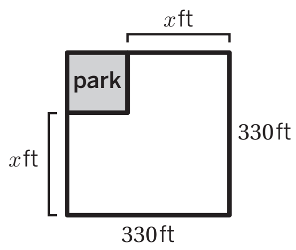
6. A square city block with a side length of 330 ft has a park located on its northwest corner, as shown in the diagram. Select the expression that represents the area of the park, in square feet.

A. $330^2 - x^2$

B. $330 - x^2$

C. $(330-x)(330-x)$

D. $330x - x$



My Notes:

SUB-UNIT

1 Connecting Quadratic Functions to Their Equations

Narrative Connections

How did the Nile River spur on Egyptian mathematics?

Egypt was one of the great civilizations of the ancient world. They gave the world the Sphinx, the pyramids, and one of the earliest systems of writing. But why Egypt?

The answer: location, location, location!

Every year, Egypt's 4,000-mile Nile River flooded, depositing layers of rich silt over the river valley and making the land fertile for crops. It also formed an expansive transportation system, allowing Egyptians to trade goods and maintain diplomatic relationships with their neighbors.

So what does this have to do with math?

Well, Egyptians were taxed based on how much land they owned — that is, the *area* of their land. Because the Nile kept flooding, this area kept changing from one year to the next. To track these changes, the Pharaoh sent surveyors to calculate the dimensions of each plot of land.

As you know, a rectangle's area is calculated by multiplying its length and width. But the land plots along the Nile rarely stayed perfectly rectangular for long. Due to the river's powerful terrain-changing forces, Egyptian surveyors used special reference tables that listed the areas of different shapes, according to different side-lengths. Using these tables, the surveyor could estimate the new area.

And so began a mathematical story of how to multiply, compute areas, and solve quadratic equations that continues to this day.

Unit 6 | Lesson 2

When and Why Do We Write Quadratic Equations?

Let's solve some quadratic equations.

Warm-up Just Right

Noah wants to purchase a larger bed for his bedroom. For the bed to fit, the width of the mattress cannot exceed 70 in. The perimeter of a mattress is given by $2w + 160$, where w is the width of the mattress, in inches.

1. A king-size mattress has a perimeter of 312 in. What is its width? Show your thinking.

2. A queen-size mattress has a perimeter of 280 in. What is its width? Show your thinking.

3. Should Noah purchase a king-size mattress or a queen-size mattress? Explain your thinking.

Log in to Amplify Math to complete this lesson online.

Name: ______________________ Date: __________ Period: __________

Activity 1 The Perfect Shot, Revisited

You previously studied the height of a ball modeled as a function of time after it was thrown into the air. The function $f(t) = -16t^2 + 80t + 64$ models the height of a ball in feet, t seconds after being launched from a mechanical device.

1. What equation could be used to determine the time the ball hits the ground?

2. Use any method, other than graphing, to determine a solution to this equation.

Reflect: How did this activity play to your strengths?

Activity 2 Revenue From Ticket Sales

The table shown represents the revenue a school is expected to earn from selling raffle tickets at p dollars each, which is modeled by the function $f(p) = -5p(p - 40)$.

p	$0	$5	$10	$15	$20	$25	$30	$35	$40
$f(p)$	$0	$875	$1,500	$1,875	$2,000	$1,875			

1. Complete the table.

2. At what ticket price(s) would the school earn $0 revenue from raffle ticket sales? Explain or show why these prices are the zeros of the function $f(p)$.

3. What does the Zero Property of Multiplication tell you about the factors of the *quadratic equation*, $-5p(p - 40) = 0$ (expressed in factored form)?

4. The school staff noticed that there are two ticket prices that each result in a revenue of $500. How would you determine these two prices?

Are you ready for more?

Using the function from Activity 2, determine the following prices without graphing.

1. If the school charges $4 per ticket, it is expected to earn $720 in revenue. Determine another price that would generate $720 in revenue.

2. If the school charges $28 per ticket, it is expected to earn $1,680 in revenue. Determine another price that would generate $1,680 in revenue.

STOP

Name: ____________ Date: ____________ Period: ____________

Summary

In today's lesson . . .

You revisited quadratic expressions in standard and factored form to help you determine the solution to a quadratic equation. In general, a ***quadratic equation*** is an equation that can be expressed in the form $ax^2 + bx + c = 0$, where a does not equal 0.

You can solve quadratic equations using a table or a graph, but it can be difficult to determine an exact answer. When trying to solve a quadratic equation algebraically, your first instinct may be to apply the properties of equality (as with a linear equation), but it is difficult to isolate the variable in this way.

In Unit 5, you learned that the zeros of a quadratic function can be identified when the quadratic is in factored form. When a quadratic equation is written in the form $ax^2 + bx + c = 0$, the zeros of $ax^2 + bx + c = 0$ are the solutions to the quadratic equation. So, determining the zeros of a quadratic expression is another strategy for solving quadratic equations.

Writing a quadratic equation in factored form is often helpful. If the product of two factors is 0, one of the factors must equal 0 due to the Zero Property of Multiplication. In the coming lessons, you will see how this helps you solve quadratic equations algebraically.

Reflect:

Practice

Name: .... Date: .... Period: ....

1. What are the zeros of the function $f(x) = (x - 5)(7x - 21)$? Select *all* that apply. (Hint: Use graphing technology.)

   A. -7
   B. -5
   C. -3
   D. 0
   E. 3
   F. 5
   G. 7

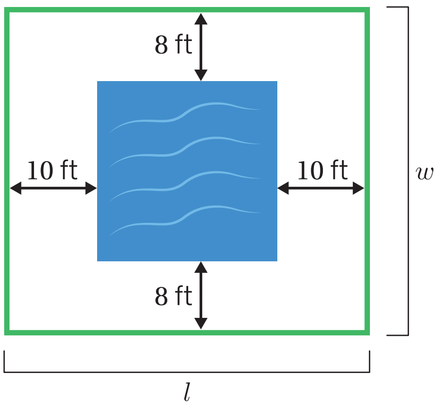
2. The two functions $f(x) = 30x^2 - 105x - 60$ and $g(x) = (5x - 20)(6x + 3)$ are equivalent.

   a. Which function would you choose to use to evaluate when $x = 0$? Explain your thinking.

   b. Determine the value of the function when $x = 0$.

3. A band is traveling to a new city to perform a concert. The revenue from their ticket sales is a function of the ticket price in dollars x, and can be modeled as the function $r(x) = (x - 6)(250 - 5x)$. Write an equation that represents the ticket price(s) for which the band should expect to earn no revenue at all.

4. The design of a square decorative pool, surrounded by a walkway, is shown. The walkway is 8 ft wide on two opposite sides of the pool, and 10 ft wide on the other two opposite sides. The final design for the pool and walkway covers a total area of $1{,}440$ ft^2. If the side length of the square pool is x, write an expression that represents each of the following.



   a. The total length of the rectangle (including the pool and walkway).

   b. The total width of the rectangle (including the pool and walkway).

   c. The total combined area of the pool and walkway.

Name: .... Date: .... Period: ....

Practice

d. Write an equation that represents a total combined area of $1{,}440$ ft^2. What does a solution to this equation mean in this context?

5. The dimensions of a screen on a tablet measure 8 in. by 5 in. The frame around the screen has a width of x in.

a. Write an expression that represents the total area of the tablet, including the frame.

b. Write an equation that represents a total area of 50.3125 in^2. Explain what a solution to this equation means in this context.

6. Determine whether each quadratic equation has a solution. If so, write the solution(s). If not, briefly explain why the equation does not have a solution.

a. $x^2 = 49$

b. $x^2 = 19$

c. $x^2 = -36$

d. $(x)(-x) = 25$

Unit 6 | Lesson 3

Solving Quadratic Equations by Reasoning

Let's use square roots to solve some quadratic equations.

Warm-up Math Talk

What strategies would you use to evaluate or solve each of the following? Use your strategy to determine the solution. Explain your thinking.

1. Evaluate 12^2

Strategy:

Solution:

2. Evaluate -12^2

Strategy:

Solution:

3. Evaluate $(-12)^2$

Strategy:

Solution:

4. Solve $x^2 = 144$

Strategy:

Solution:

Log in to Amplify Math to complete this lesson online.

Name: ________________ Date: ________ Period: ________

Activity 1 How Many Solutions?

Plan ahead: How will you stay focused on the task while using graphing technology?

Several quadratic equations are shown. For each equation, complete the table to determine the solution(s) and then state the number of solutions. Be prepared to explain your thinking.

Equation	Solution(s)	Number of solutions
$x^2 = 9$		
$x^2 = 0$		
$x^2 - 1 = 3$		
$2x^2 = 50$		
$(x+1)(x+1) = 0$		

Are you ready for more?

How many solutions does the equation $(x-1)(x-1) = 4$ have? What are they? Explain or show your thinking.

Activity 2 Determining Pairs of Solutions

Each of the following equations has two solutions. Determine the two solutions. Explain or show your thinking.

1. $n^2 + 4 = 404$

2. $432 = 3n^2$

3. $144 = (n + 1)^2$

4. $(n - 5)^2 - 30 = 70$

Are you ready for more?

1. How many solutions does the equation $(x - 3)(x + 1)(x + 5) = 0$ have? What are they?

2. How many solutions does the equation $(x - 2)(x - 7)(x - 2) = 0$ have? What are they?

3. Write your own equation that has 10 solutions.

Name: ______ Date: ______ Period: ______

Activity 3 I Have . . . Who Has?

You will play the game "I Have . . . Who Has?" to match quadratic equations with their solution(s). You will be given cards and will need a sheet of paper and a pencil. Please attend carefully to the instructions.

Rules:

- Play begins with the card that says, "This is the first card."
- Whoever has this card reads the "I have the equation ______, who has the solution?" question aloud. Then they write the equation on the board.
- Everyone else works to determine the solution(s) to the equation.
- Raise your hand if you have the solution on your card.
- Explain why your card is the solution.
- Read the question on your card aloud and write the equation on the board.
- Repeat until there is one card remaining that says, "This is the last card."

STOP

Summary

In today's lesson . . .

You solved quadratic equations by reasoning about values that make an equation true. By applying the properties of equality, you were able to rearrange equations so that you could determine the square root of square integers, which is another strategy for solving quadratic equations. When solving this way, you used the ***plus-or-minus*** solutions to the square root of square integers. You also saw examples of quadratic equations with two solutions. Because positive square integers have two solutions, it is possible that when solving a quadratic equation, it will also produce two solutions because you must account for both square root values.

Reflect:

Practice

Name: ________ Date: ________ Period: ________

1. Consider the quadratic equation $x^2 = 9$.

a Show that 3, -3, $\sqrt{9}$, and $-\sqrt{9}$ are all solutions to the equation.

b Show that 9 and $\sqrt{3}$ are *not* solutions to the equation.

2. Solve the quadratic equation $(x-1)^2 = 16$. Explain or show your thinking.

3. The table shows one way to solve the quadratic equation $\frac{5}{9}y^2 = 5$. Describe what happens in each step.

	Describe the step:
$\frac{5}{9}y^2 = 5$	**Step 1:** Write the original equation.
$5y^2 = 45$	**Step 2:**
$y^2 = 9$	**Step 3:**
$y = 3$ or $y = -3$	**Step 4:**

4. A set of kitchen containers can be stacked to save space. The height of the stack is given by the expression $(1.5c + 7.6)$ cm, where c is the number of containers.

a Determine the height of a stack that has 8 containers.

b A tower created with containers is 40.6 cm tall. How many containers are in the tower?

c Noah looks at the expression and says, "7.6 must be the height of a single container." Do you agree with Noah? Explain your thinking.

Practice

Name: ________ Date: ________ Period: ________

5. As part of a publicity stunt, a TV host drops a watermelon from the top of a tall building. The height of the watermelon t seconds after it is dropped is given by the function $h(t) = 850 - 16t^2$, where h is measured in feet.

a Determine $h(4)$. Explain the meaning of this value in this context.

b Determine $h(0)$. What does this value tell you about the watermelon and the building?

c Is the watermelon still in the air 8 seconds after it is dropped? Explain your thinking.

6. Solve each equation. Show your thinking.

a $2x - 4 = 8$

b $-2(x - 4) = 8$

c $\frac{1}{3}(6x - 12) + 2x = 8$

d $2x + 1 = 4x - 3$

Name: .... Date: .... Period: ....

Unit 6 | Lesson 4

The Zero Product Principle

Let's determine solutions of equations when products equal 0.

Warm-up Algebra Talk

Discuss the strategies you would use to determine which values make each of the following equations true. Then determine the solution to each equation.

1. $6 + 2a = 0$

Strategy:

Solution:

2. $7(c - 5) = 0$

Strategy:

Solution:

3. $7b = 0$

Strategy:

Solution:

4. $d \cdot e = 0$

Strategy:

Solution:

Log in to Amplify Math to complete this lesson online.

Activity 1 Solving Quadratics Algebraically

One partner will complete Column A and the other will complete Column B. Complete the problems in your column, and then compare responses with your partner. Discuss and resolve any differences.

1. For each column, determine the solution(s) to each equation. Be prepared to explain your thinking.

Column A	Column B
$x - 3 = 0$	$2x + 11 = 0$
$x + 11 = 0$	$x(2x + 11) = 0$
$(x - 3)(x + 11) = 0$	$(x - 3)(2x + 11) = 0$

2. What do you notice about the solutions of each equation?

3. How many solutions does the equation $x(x + 3)(3x - 4) = 0$ have? Explain or show your thinking.

Are you ready for more?

Consider the quadratic equation $(x - 3)(x + 5) = 48$.

1. Use factors of 48 to determine as many solutions to the equation as you can.

2. Once you think you have all the solutions, explain why these must be the *only* solutions.

Name: ________ Date: ________ Period: ________

Activity 2 Revisiting Projectiles

The following functions are equivalent and approximate the height of a certain projectile in meters, t seconds after launch.

$$h(t) = -5t^2 + 27t + 18 \qquad k(t) = (-5t - 3)(t - 6)$$

1. Which function provides the best use of the ***Zero Product Principle***? Explain your thinking.

2. What information can you determine by using the Zero Product Principle in this context? Explain your thinking.

3. Without graphing, use the Zero Product Principle to determine the information you mentioned in your response to Problem 2. Show your thinking.

STOP

Summary

In today's lesson . . .

You learned about the ***Zero Product Principle***, which states that if the product of two factors is 0, then one or both of the factors must be 0. In other words, if $a \bullet b = 0$, then $a = 0$, $b = 0$, or both are equal to 0.

This property is helpful when solving quadratic equations, especially if they can be written in factored form as a product of expressions that are equal to zero. You can determine the solutions by setting each factor equal to zero and solving those equations.

Reflect:

Practice

Name: ________ Date: ________ Period: ________

1. If the quadratic equation $(x + 10)x = 0$ is true, which statement must also be true, according to the Zero Product Principle?

   A. Only $x = 0$

   B. Only $x + 10 = 0$

   C. Either $x^2 = 0$ or $10x = 0$

   D. Either $x = 0$ or $x + 10 = 0$

2. What are the solutions to the quadratic equation $(10 - x)(3x - 9) = 0$?

   A. 10 and 3

   B. 10 and 9

   C. -10 and 3

   D. -10 and 9

3. Solve each quadratic equation. Show or explain your thinking.

   a. $(x - 6)(x + 5) = 0$

   b. $(x - 3)\left(\frac{2}{3}x - 6\right) = 0$

   c. $(-3x - 15)(x + 7) = 0$

4. Select *all* the expressions that are equivalent to $4(2 + 3x)$.

   A. $8 + 3x$

   B. $4(5x)$

   C. $8 + 12x$

   D. $12x + 2$

   E. $12x + 8$

   F. $2(4) + 3x(4)$

   G. $2(2 + 3x) + 2(2 + 3x)$

Practice

Name: .... Date: .... Period: ....

5. Each expression represents the area of a rectangle. Determine a possible length and width of each rectangle. Explain or show your thinking.

a $3x + 21$

b $4(9) + 4(20)$

c $8^2 + 8a$

6. Mai wants to determine the zeros of the quadratic function $f(x) = x^2 - 25$. Select *all* of the true statements.

A. Mai is looking for all of the input values of the function.

B. Mai is looking for the value of $f(0)$.

C. Mai wants to determine the value of x when $f(x) = 0$.

D. Mai can determine the zeros by solving the equation $x^2 - 25 = 0$.

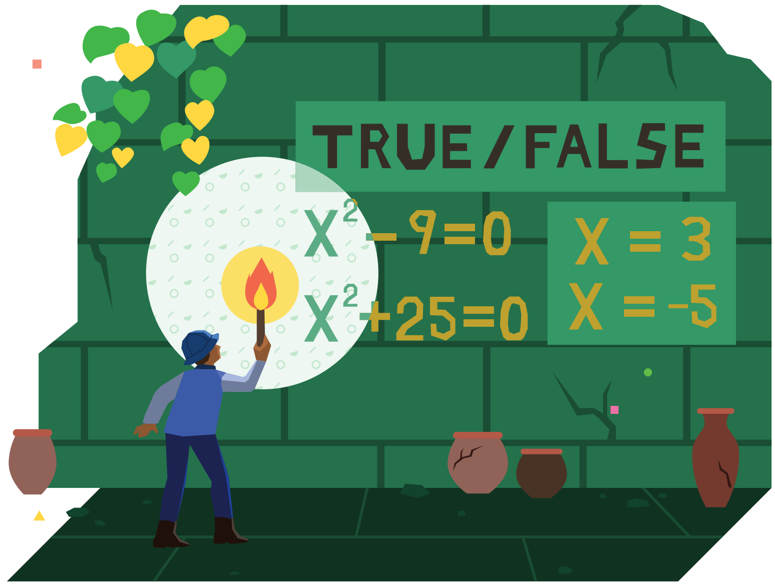
E. Mai can determine the zeros by graphing the function and determining the x-intercepts.

Name: ______ Date: ______ Period: ______

Unit 6 | Lesson 5

How Many Solutions?

Let's use graphs to investigate the number of solutions to a quadratic equation.



Warm-up Four Equations

Determine whether each statement is true or false.

Statement	True or False?	Explain or show your thinking.
3 is the only solution to the equation $x^2 - 9 = 0$.		
A solution to the equation $x^2 + 25 = 0$ is -5.		
The equation $x(x - 7) = 0$ has two solutions.		
5 and -7 are the solutions to the equation $(x - 5)(x + 7) = 12$.		

Log in to Amplify Math to complete this lesson online.

Activity 1 Solving by Graphing

Study these equations.

Set A	Set B
$(x - 5)(x - 3) = 0$	$(x - 5)(x - 3) = -1$
$(x - 5)(x - 3) + 1 = 0$	$(x - 5)(x - 3) = -4$
$(x - 5)(x - 3) + 4 = 0$	

1. What do you notice about the equations in Set A? Set B?

2. Which equation(s) can you solve using what you already know about the factored form of quadratic equations? Explain your thinking. Solve the equation(s).

3. Han and Lin use different strategies to solve the equation $(x - 5)(x - 3) + 1 = 0$. Study each person's strategy.

Han's strategy:

$(x - 5)(x - 3) + 1 = 0$

I graphed the equation $y = (x - 5)(x - 3) + 1$ and found there is one zero, $x = 4$. So, there is one solution, 4.

Lin's strategy:

$(x - 5)(x - 3) + 1 = 0$

$(x - 5)(x - 3) = -1$

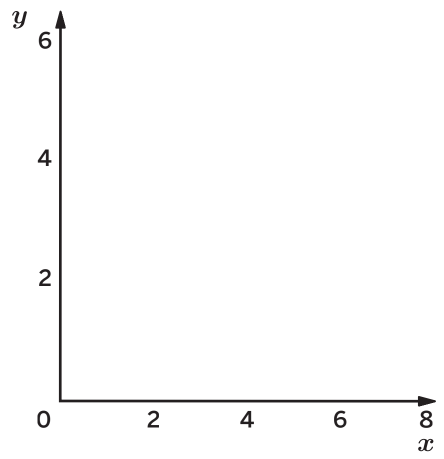
I graphed the equation $y = (x - 5)(x - 3)$ and found the zeros are $x = 5$ and $x = 3$. So, the solutions are 5 and 3.

Which strategy is correct? Explain your thinking.

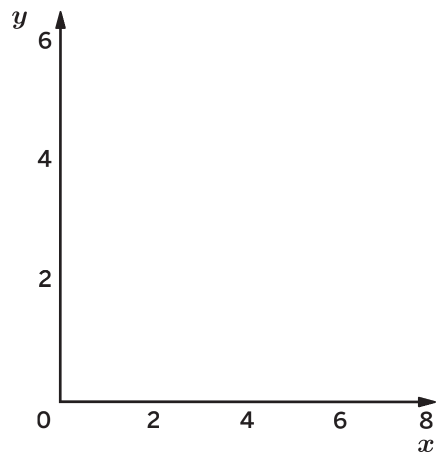
Name: ______ Date: ______ Period: ______

Activity 1 Solving by Graphing (continued)

4. Use the correct strategy from Problem 3 to solve the equation $(x-5)(x-3)+1=0$. Draw a sketch of your graph here. Identify the solution(s). How many solutions are there?



5. Use the correct strategy from Problem 3 to try to solve the equation $(x-5)(x-3)+4=0$. Draw a sketch of your graph here. What do you notice?



6. For a quadratic equation set equal to 0, make a conjecture about how the number of x-intercepts is related to the number of solutions to the equation.

Plan ahead: How will you stay focused on the task while using graphing technology?

Activity 2 Determining all the Solutions

1. Solve each equation. Be prepared to explain your thinking.

a. $x^2 = 121$

b. $x^2 - 31 = 5$

c. $(x - 4)(x - 4) = 0$

d. $(x + 3)(x - 1) = 5$

e. $x^2 = -25$

f. $(x - 4)(x - 1) = 990$

g. $(x + 7)^2 = 0$

h. $(x + 1)^2 = -4$

i. $(x - 8)(x + 3) = 0$

Are you ready for more?

The equations $(x - 3)(x - 5) = -1$, $(x - 3)(x - 5) = 0$, and $(x - 3)(x - 5) = 3$ all have whole number solutions.

1. Use graphing technology to graph each of the following pairs of equations on the same coordinate plane. How can you use the two graphs to solve the related equations?

   a. $y = (x - 3)(x - 5)$ and $y = -1$

   b. $y = (x - 3)(x - 5)$ and $y = 0$

   c. $y = (x - 3)(x - 5)$ and $y = 3$

2. Use the graphs to help you determine three more equations of the form $(x - 3)(x - 5) = z$ that have whole number solutions, where z is a constant.

3. Determine a pattern in the values of z that give whole number solutions.

4. Without solving, determine if $(x - 3)(x - 5) = 120$ and $(x - 3)(x - 5) = 399$ have whole number solutions. Explain your thinking.

Name: .... Date: .... Period: ....

Activity 3 Find and Fix

1. Priya considers the quadratic equation $(x-5)(x+1)=7$. She reasons that if this equation is true, then either $(x-5)=7$ or $(x+1)=7$, so $x=12$ or $x=6$ are solutions to the original equation. Do you agree? If so, explain your thinking. If not, explain the mistake in Priya's thinking.

2. Diego and Mai consider the quadratic equation $x^2-10x=0$. Study each person's strategy used to solve the equation.

Diego's Strategy	Mai's Strategy
Work:	**Work:**
$x^2-10x=0$	$x^2-10x=0$
$x(x-10)=0$	$x(x-10)=0$
$x-10=0$	$x=0$ or $x=10$
$x=10$	
Explanation:	**Explanation:**
• Rewrite in factored form.	• Rewrite in factored form.
• Divide each side by x.	

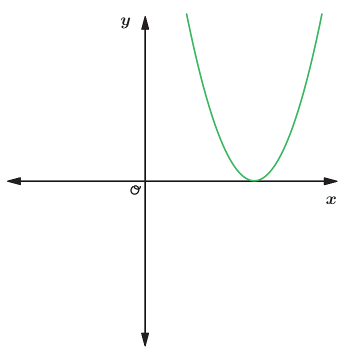
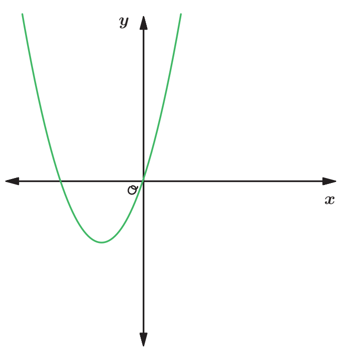
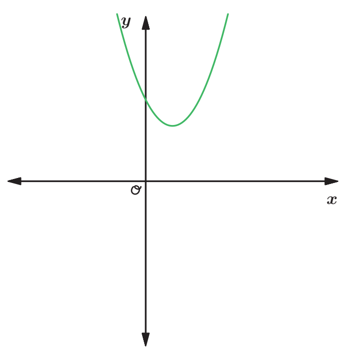
Do you agree with either strategy? Explain your thinking.

STOP

Summary

In today's lesson . . .

You saw that quadratic equations can have two, one, or no solutions. The number of solutions a quadratic equation has can be found by rearranging the equation so that one side is equal to 0. Then you can graph the quadratic equation and count how many x-intercepts it has. Each x-intercept represents a solution to the quadratic equation.

One solution	Two solutions	No solutions
One x-intercept	**Two x-intercepts**	**No x-intercepts**
		

Today, you saw quadratic equations whose solutions happened to be whole numbers, but this will not always be the case. While graphing can tell you *how many* solutions there are, you cannot always solve for the precise values of those solutions with a graph. This means you still need algebraic ways of solving for exact solutions. And remember, when you are solving a quadratic equation algebraically, avoid dividing both sides by the same variable, because that may eliminate a solution.

Reflect:

Name: ______ Date: ______ Period: ______

Practice

1. How would you rewrite each equation so that you could determine its solution by graphing?

a) $3x^2 = 81$

b) $(x-1)(x+1) - 9 = 5x$

c) $x^2 - 9x + 10 = 32$

d) $6x(x-8) = 29$

2. Consider the three quadratic functions $f(x) = x^2 + 4$, $g(x) = x(x+3)$, and $h(x) = (x-1)^2$.

a) Sketch a graph for each function, either by hand or using technology.

$f(x) = x^2 + 4$	$g(x) = x(x+3)$	$h(x) = (x-1)^2$

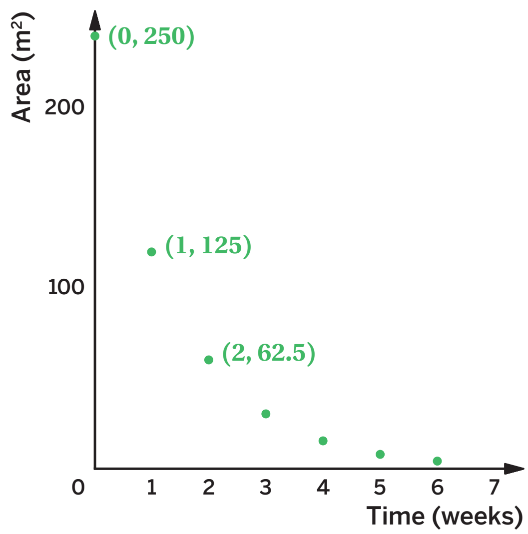
b) Determine the number of solutions to each equation: $f(x) = 0$, $g(x) = 0$, and $h(x) = 0$. Explain your thinking.

Practice

Name: .... Date: .... Period: ....

3. Mai solves the equation $(x - 5)^2 = 0$. She determines that the solutions are $x = 5$ and $x = -5$. Han disagrees with her solutions. He says that $x = 5$ is the only solution. Do you agree with Mai or Han? Explain your thinking.

4. The graph shows the area A, in square meters, covered by algae in a lake w weeks after it was first measured. In a second lake, the area B, in square meters, covered by algae after w weeks is given by the equation $B = 975 \cdot \left(\frac{2}{5}\right)^w$. For which algae population is the area decreasing more rapidly? Explain your thinking.



5. If the equation $(x - 4)(x + 6) = 0$ is true, which statement is also true according to the Zero Product Principle?

A. Only $x - 4 = 0$

B. Only $x + 6 = 0$

C. $x = -4$ or $x = 6$

D. $x - 4 = 0$ or $x + 6 = 0$

6. Match the equivalent expressions:

a $-9 \cdot (-8) \div (-1)$	....	$-(-(-6))$
b $-1 \cdot 6$	....	$-(2 \cdot (-1))$
c $-1 - 6$	....	$4 \cdot (-3) \cdot 6$
d $-2(3 - 4)$	....	$-6 - (-6)$
e $4 \div (-1)$	....	$2 \cdot (-2)$
f $6 + (-2)(-3)$	....	$-1 + (-6)$

SUB-UNIT

2 Factoring Quadratic Expressions and Equations

Narrative Connections

When is zero more than nothing?

There are few numbers as elegant as the humble zero — that unique number you can add to any number without changing its value. And yet, it took humanity a long time to even imagine such a number could exist!

Several ancient civilizations, including the Mesopotamians and the Mayans, had symbols that were *kind of like* a zero. These symbols were used as placeholders, making it easy to distinguish 102 from 12, for example.

But to see how zero became an actual *number*, we journey to 7th century India. Here, the mathematician Brahmagupta laid out the mathematical properties of zero, noting that a positive number subtracted from itself equaled this new special value!

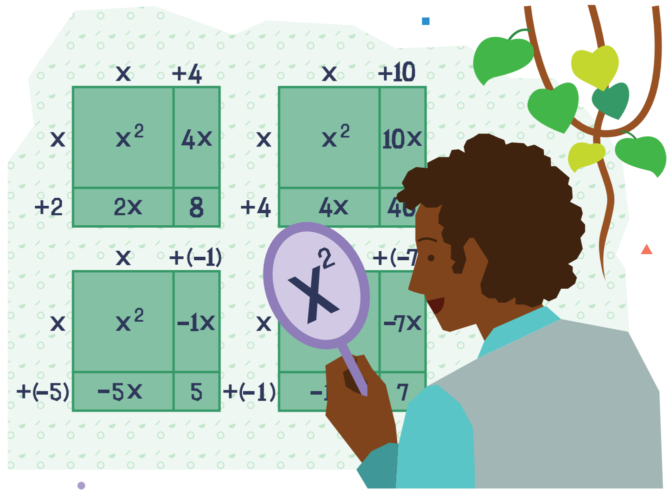
There's debate as to precisely how zero emerged in India. Some scholars think it's related to Hindu and Buddhist philosophy, where the idea of spiritual "nothingness" was easier to grasp. Whatever the case may be, the idea took hold and spread to China and the Middle East. But as Arab traders traveled into Europe, the idea was met with strenuous resistance.

Medieval European thinkers, ironically, equated zero with chaos. In 1299, the city of Florence even went so far as to ban the number (along with other Indo-Arabic numerals), claiming it was too easy to doctor into other numbers. It wouldn't be until the 16th century that Europe finally embraced the goose egg we all know and love today.

We know now that zero is more than, well, nothing. Seeing where a parabola's y-coordinate equals zero can tell you when a ball will hit the ground, when profits turn into losses, and when rectangular gardens cannot hold any flowers. What's more, writing a quadratic expression so that it is *equal to* zero can be a powerful first step toward *solving* it.

Unit 6 | Lesson 6

Writing Quadratic Expressions in Factored Form (Part 1)



Let's write quadratic expressions in factored form.

Warm-up Puzzles of Rectangles

How can you determine the missing area in Figure A? The missing length in Figure B? Explain your thinking.

Figure A

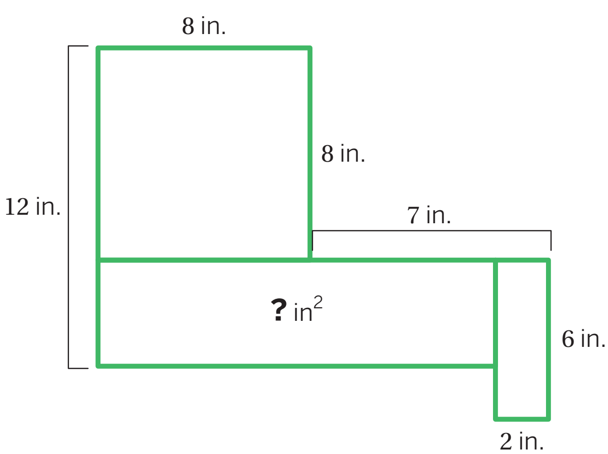
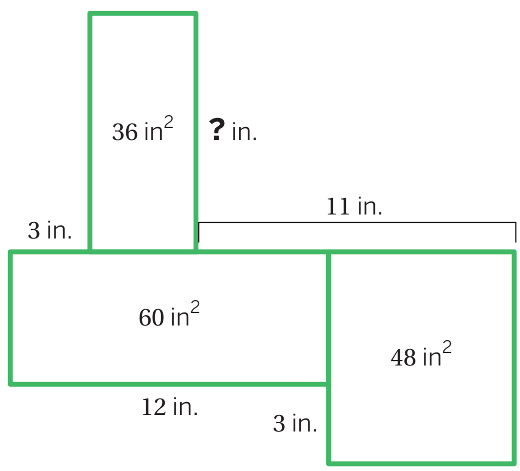


Figure B



Missing area of Figure A:

Missing length of Figure B:

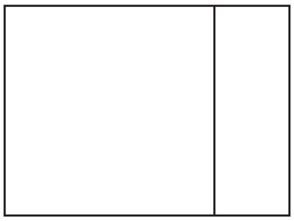
Log in to Amplify Math to complete this lesson online.

Name: ______ Date: ______ Period: ______

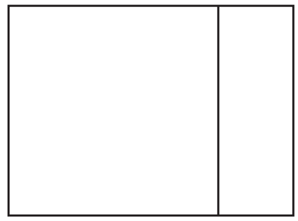
Activity 1 Using Diagrams to Understand Equivalent Expressions

Each problem shows a quadratic expression in factored form and standard form. For each problem, complete the area diagram to show that the expressions are equivalent.

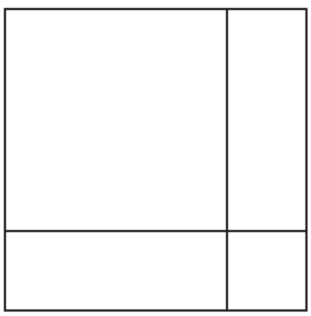
1. $x(x+3)$ and x^2+3x



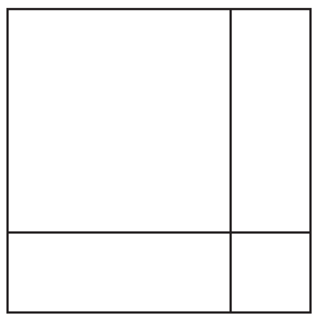
2. $x(x-6)$ and x^2-6x



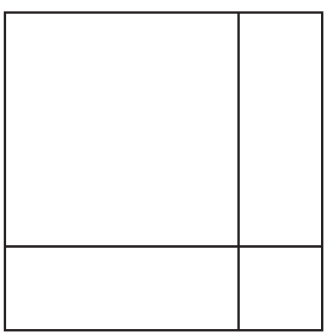
3. $(x+2)(x+4)$ and x^2+6x+8



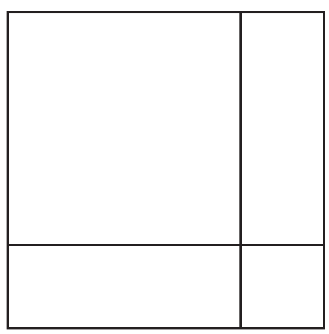
4. $(x+4)(x+10)$ and $x^2+14x+40$



5. $(x-5)(x-1)$ and x^2-6x+5



6. $(x-1)(x-7)$ and x^2-8x+7



7. Study the expressions that involve the product of *two sums* or *two differences*. How is each expression in factored form related to its equivalent expression in standard form?

Activity 2 Applying the Distributive Property

Each row in the table contains a pair of equivalent expressions. Complete the table by writing the missing equivalent expression. Consider drawing a diagram, if helpful.

Factored form	Standard form
$x(x+7)$	
	x^2+9x
	x^2-8x
$(x+6)(x+2)$	
	$x^2+13x+12$
$(x-6)(x-2)$	
	$x^2-7x+12$
	x^2+6x+9
	$x^2+10x+9$
	$x^2-10x+9$
	x^2-6x+9
	$x^2+(m+n)\,x+mn$

Are you ready for more?

A mathematician threw a party and told her guests this riddle: "I have three daughters. The product of their ages is 72. The sum of their ages is my house number. How old are my daughters?" The guests went outside to see the house number, 14. They said, "This riddle cannot be solved!" The mathematician said, "I forgot to tell you the last clue. My youngest daughter prefers strawberry ice cream."

With this last clue, the guests could solve the riddle. How old are the mathematician's daughters?

STOP

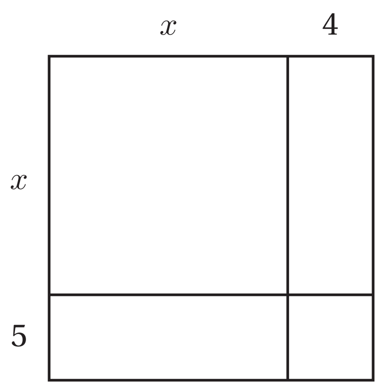
Name: .... Date: .... Period: ....

Summary

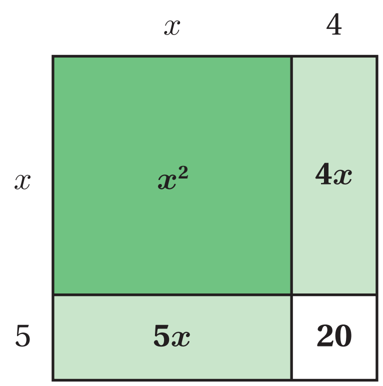
In today's lesson . . .

You used diagrams and the Distributive Property to better understand the equivalent factored and standard forms of quadratic expressions. Recall that you previously learned how to expand a quadratic expression in factored form and write it in standard form by applying the Distributive Property.

To keep track of all the products, you can create an area diagram like this:



Then write the products of each pair inside the spaces.



The area diagram shows that the expression $(x+4)(x+5)$ is equivalent to $x^2 + 5x + 4x + 20$, which is equivalent to $x^2 + 9x + 20$.

Reflect:

Practice

Name: .... Date: .... Period: ....

1. Use a diagram to show that the expressions $(x + 4)(x + 2)$ and $x^2 + 6x + 8$ are equivalent.

2. Select *all* expressions that are equivalent to the expression $x - 5$.

A. $5 + x$

B. $5 - x$

C. $-5 + x$

D. $-5 - x$

E. $x + (-5)$

F. $x - (-5)$

G. $-5 - (-x)$

3. Determine the missing values for each pair of equivalent expressions.

a x^2 .... x .... and $(x - 9)(x - 3)$

b $x^2 + 12x + 32$ and $(x + 4)(x$ .... $)$

c $x^2 - 12x + 35$ and $(x - 5)(x$ .... $)$

d $x^2 - 9x + 20$ and $(x - 4)(x$ .... $)$

Practice

Name: ______ Date: ______ Period: ______

4. Determine *all* possible values for the variable that make each equation true.

a. $b(b - 4.5) = 0$

b. $(2x + 4)(x - 4) = 0$

5. *Graphing technology required*. When solving the equation $(2 - x)(x + 1) = 11$, Priya graphs the equation $y = (2 - x)(x + 1) - 11$ and then determines where the graph intersects the x-axis. Tyler looks at her work and says that graphing is unnecessary. Tyler says that Priya can set up the equations $2 - x = 11$ and $x + 1 = 11$, so the solutions are $x = -9$ or $x = 10$.

a. Do you agree with Tyler? If not, where is the mistake in his reasoning?

b. Graph Priya's equation. How many solutions does the equation have?

6. Determine two numbers that:

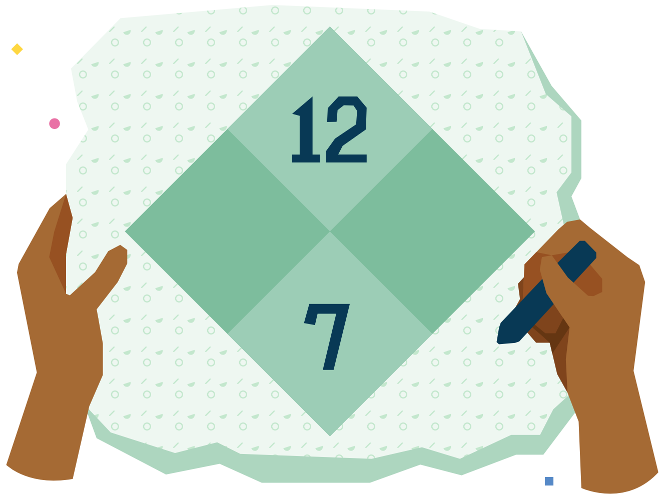
a. Have a product of 100 and a sum of 52.

b. Have a product of 50 and a sum of 15.

c. Have a product of 10 and a sum of -7.

Unit 6 | Lesson 7

Writing Quadratic Expressions in Factored Form (Part 2)

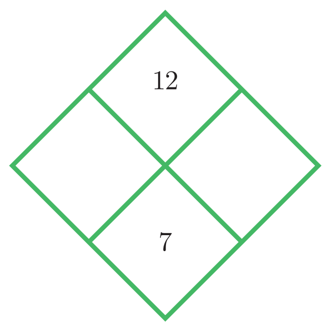


Let's write more quadratic expressions in factored form.

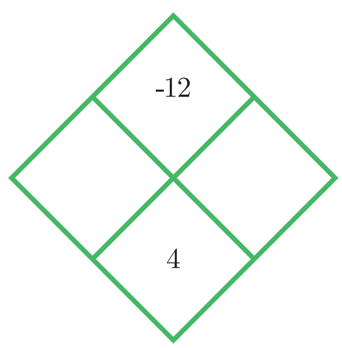
Warm-up Sums and Products

Complete each diamond to show the two numbers that have the given products and sums. The product of the two factors is the top number in the diamond. The sum of the two factors is the bottom number in the diamond.

1. Two numbers have a product of 12 and a sum of 7. Complete the diamond to show these two numbers.



2. Two numbers have a product of -12 and a sum of 4. Complete the diamond to show these two numbers.



3. List all the integer factors of 12. Which pair of factors have a sum of -7?

4. List all the integer factors of -27. Which pair of factors have a sum of 26?

Log in to Amplify Math to complete this lesson online.

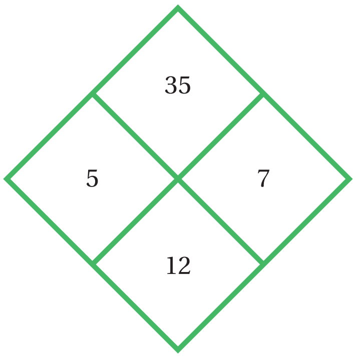
Name: ______________ Date: ______________ Period: ______________

Activity 1 Diamond Puzzles

Complete each diamond. Then use these values to complete the factored form for each quadratic expression. Problem 1 is already completed as a guide.

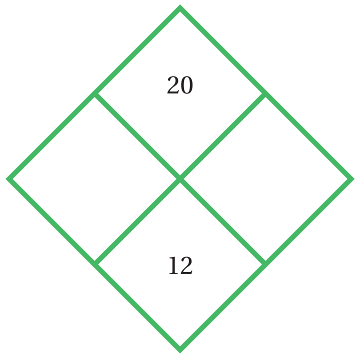
1. $x^2 + 12x + 35$

$(x$ **+ 5** $)(x$ **+ 7** $)$



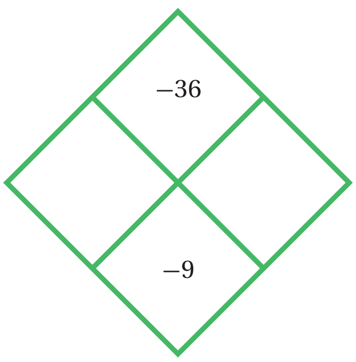
2. $x^2 + 12x + 20$

$(x$ ______ $)(x$ ______ $)$



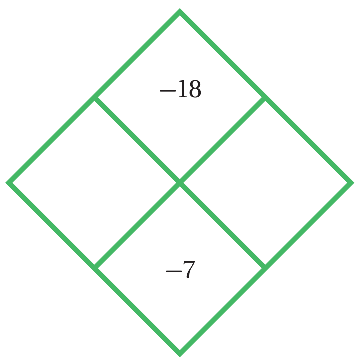
3. $x^2 - 9x - 36$

$(x$ ______ $)(x$ ______ $)$



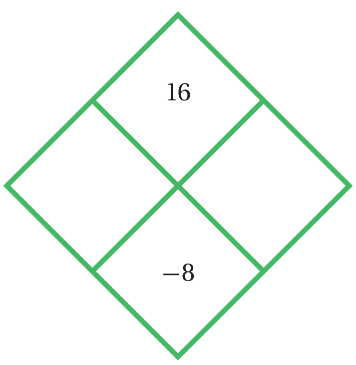
4. $x^2 - 7x - 18$

$(x$ ______ $)(x$ ______ $)$



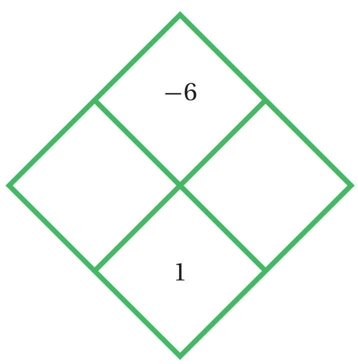
5. $x^2 - 8x + 16$

$(x$ ______ $)(x$ ______ $)$



6. $x^2 + x - 6$

$(x$ ______ $)(x$ ______ $)$



Activity 2 Negative Constants

1. Set A shows a table where each row contains a pair of equivalent quadratic expressions. Complete the table with each missing equivalent form.

Set A

Factored form	Standard form
$(x+5)(x+6)$	
	$x^2+13x+30$
$(x-3)(x-6)$	
	$x^2-11x+18$

2. Set B shows a table where each row contains a pair of equivalent quadratic expressions. Complete the table with each missing equivalent form.

Set B

Factored form	Standard form
$(x+12)(x-3)$	
	$x^2-9x-36$
	$x^2-35x-36$
	$x^2+35x-36$

3. How do the expressions in Set B differ from the expressions in Set A? Explain your thinking.

Stronger and Clearer: Share your responses to Problem 3 with another pair of students and make any revisions. How do the structures of the expressions compare? What math language can you use?

Name: ________ Date: ________ Period: ________

Activity 3 Factors of 100 and -100

1. Consider the quadratic expression $x^2 + bx + 100$.

a Complete the tables so that:

- The first table shows all factor pairs of 100 that result in positive values of b.
- The second table shows all factor pairs of 100 that result in negative values of b.
- Use as many rows as needed.

b Add each factor pair to determine b.

Positive value(s) of b

Factor 1	Factor 2	b

Negative value(s) of b

Factor 1	Factor 2	b

2. Consider the quadratic expression $x^2 + bx - 100$.

a Complete the tables so that:

- The first table shows all factor pairs of -100 that result in positive values of b.
- The second table shows all factor pairs of -100 that result in negative values of b.
- The third table shows all factor pairs of -100 that result in a value of b of 0.
- Use as many rows as needed.

b Add each factor pair to determine b.

Positive value(s) of b

Factor 1	Factor 2	b

Negative value(s) of b

Factor 1	Factor 2	b

Zero value(s) of b

Factor 1	Factor 2	b

Activity 3 Factors of 100 and −100 (continued)

3. Use the tables in Problems 1 and 2 to write each quadratic expression in factored form.

a. $x^2 - 25x + 100$

b. $x^2 + 15x - 100$

c. $x^2 - 15x - 100$

d. $x^2 + 99x - 100$

Are you ready for more?

How many different integer values of c can you find so that the quadratic expression $x^2 + 10x + c$ can be written in factored form?

STOP

Name: ______________________ Date: __________ Period: __________

Summary

In today's lesson . . .

You further explored the structure of equivalent quadratic expressions written in factored form and standard form. When you write quadratic expressions in factored form, it is helpful to remember:

- Multiplying two positive numbers or two negative numbers results in a positive product.
- Multiplying a positive number and a negative number results in a negative product.

For example, if you want to determine two factors whose product is 10, the factors must either both be positive or both negative. If you want to determine two factors whose product is -10, one of the factors must be positive and the other one must be negative.

Reflect:

Practice

Name: .... Date: .... Period: ....

1. Match each quadratic expression in standard form with its equivalent expression in factored form.

Standard form	Factored form
a $x^2 - 2x - 35$	.... $(x + 5)(x + 7)$
b $x^2 + 12x + 35$	.... $(x - 5)(x - 7)$
c $x^2 + 2x - 35$	.... $(x + 5)(x - 7)$
d $x^2 - 12x + 35$	.... $(x - 5)(x + 7)$

2. Determine two numbers that:

a Have a product of -40 and a sum of -6.

b Have a product of -40 and a sum of 6.

c Have a product of -36 and a sum of 9.

d Have a product of -36 and a sum of -5.

3. Determine two numbers that:

a Have a product of 17 and a sum of 18.

b Have a product of 20 and a sum of 9.

c Have a product of 11 and a sum of -12.

d Have a product of 36 and a sum of -20.

Practice

Name: ________ Date: ________ Period: ________

4. Rewrite each quadratic expression in factored form. Use a diagram, if helpful.

a. $x^2 - 3x - 28$

b. $x^2 + 3x - 28$

c. $x^2 - 12x - 28$

d. $x^2 - 28x - 60$

5. Consider the function $p(x) = \dfrac{x-3}{2x-6}$.

a. Evaluate $p(1)$. Show or explain your thinking.

b. Evaluate $p(3)$. Show or explain your thinking.

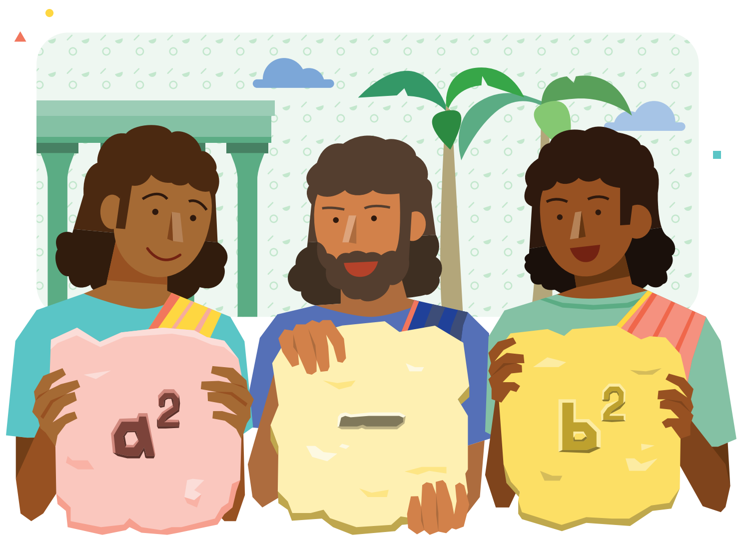
c. What is the domain of p?

6. Determine the product $(x - 10)(x + 10)$. How do you think the terms in the product relate to the terms in the factors?

Unit 6 | Lesson 8

Special Types of Factors

Let's study special types of factors.



Warm-up Math Talk

What strategies could be used to determine the value of this expression? Be prepared to explain your thinking.

$$\frac{(54^2 - 29^2)}{54 + 29}$$

Critique and Correct: Your teacher will display an incorrect strategy for determining the value of the expression. Identify the flaw in the reasoning behind this strategy and explain why it is incorrect.

Log in to Amplify Math to complete this lesson online.

Name: ______ Date: ______ Period: ______

Activity 1 Deriving Difference of Squares

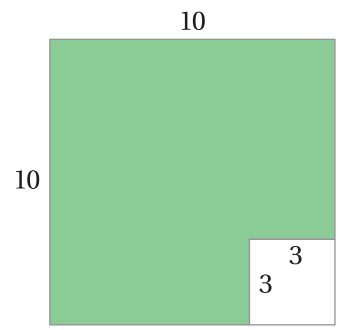
1. Refer to Figure A. All measurements are in inches.

a What is the area of the large square?

b What is the area of the small square?

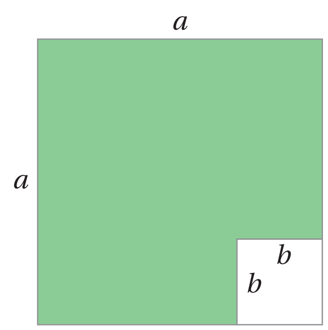
c What is the area of the figure if the small square is removed?

Figure A



2. Refer to Figure B. Write an expression to represent the area of the figure if the small square is removed.

Figure B

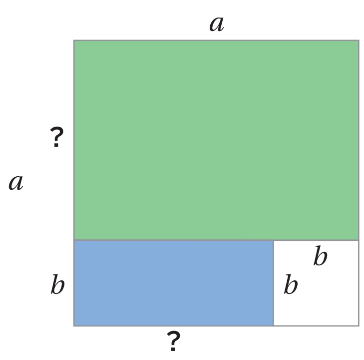


3. Refer to Figure C.

a Determine the missing side lengths of the figure.

b The small square with side length b is removed. The rectangle is rotated and moved to form a new rectangle. Sketch this new figure and label its side lengths.

Figure C



c Write an expression for the area of the new figure.

d Show that the expressions in Problem 2 and part c of Problem 3 are equivalent.

Activity 2 Writing Products as Differences

1. Clare claims that the expression $(10 + 3)(10 - 3)$ is equivalent to $10^2 - 3^2$ and that the expression $(20 + 1)(20 - 1)$ is equivalent to $20^2 - 1^2$. Do you agree? Show your thinking.

2. Use your observations from Problem 1 to evaluate the expression $(100 + 5)(100 - 5)$. Verify your response by calculating $105 \cdot 95$. Show your thinking.

3. Is the expression $(x + 4)(x - 4)$ equivalent to $x^2 - 4^2$? Support your response with and without a diagram.

   With a diagram: **Without a diagram:**

4. Is the expression $(x + 4)^2$ equivalent to $x^2 + 4^2$? Support your response with and without a diagram.

   With a diagram: **Without a diagram:**

Name: ______ Date: ______ Period: ______

Activity 3 When There Is No Linear Term

Complete the table to show an equivalent expression for each form. One row does not have an equivalent form.

Factored form	Standard form
$(x-10)(x+10)$	
$(2x+1)(2x-1)$	
$(4-x)(4+x)$	
	x^2-81
	$49-y^2$
	$9z^2-16$
	$25t^2-81$
$\left(c+\frac{2}{5}\right)\left(c-\frac{2}{5}\right)$	
	$\frac{49}{16}-d^2$
$(x+5)(x+5)$	
	x^2-6
	x^2+100

STOP

Summary

In today's lesson . . .

You explored quadratic expressions that do not have linear terms in standard form. In general, a quadratic expression that is a ***difference of squares***, such as $a^2 - b^2$, can be factored as $(a + b)(a - b)$.

Reflect:

Name: .... Date: .... Period: ....

Practice

1. Match each quadratic expression in factored form with an equivalent expression in standard form. One expression in standard form has *no* match.

Factored form	**Standard form**
a $(y+x)(y-x)$	.... $121-x^2$
b $(11+x)(11-x)$	.... $x^2+2xy-y^2$
c $(x-11)(x+11)$	.... y^2-x^2
d $(x-y)(x-y)$	.... $x^2-2xy+y^2$
	.... x^2-121

2. Both the expressions $(x-3)(x+3)$ and $(3-x)(3+x)$ contain a sum and a difference, with only the terms 3 and x in each factor. If each expression is rewritten in standard form, will the two standard expressions be equivalent? Show your thinking.

3. Use what you know about the difference of squares to complete these problems.

   a. Show that the expressions $(5+1)(5-1)$ and 5^2-1^2 are equivalent.

   b. The expressions $(30-2)(30+2)$ and 30^2-2^2 are equivalent and can be used to determine the product of two numbers. What are these two numbers?

   c. Write $94 \cdot 106$ as a product of a sum and a difference, and then as a difference of squares. Use your expression to calculate the value of $94 \cdot 106$.

Practice

Name: ________ Date: ________ Period: ________

4. Draw a diagram to show that the expression $(x - 3)(x - 7)$ is equivalent to the expression $x^2 - 10x + 21$.

5. Select *all* the expressions that are equivalent to the expression $8 - x$.

A. $x - 8$

B. $-8 + x$

C. $-x + 8$

D. $8 + (-x)$

E. $x + (-8)$

F. $x - (-8)$

G. $-x - (-8)$

6. Factor the expression $x^2 - x - 12$.

Name: ________ Date: ________ Period: ________

Unit 6 | Lesson 9

Solving Quadratic Equations by Factoring

Let's solve some quadratic equations without graphing.

Warm-up What Is That Number?

Find at least one solution to the equation $x^2 - 2x - 35 = 0$ using the following steps.

1. Choose a whole number between 0 and 10.

2. Evaluate the expression $x^2 - 2x - 35$, substituting your whole number for the value of x.

3. If substituting your number did not give a value of 0, find someone in your class who may have chosen a number that *does* give a value of 0 for the expression. Which number is it?

4. There is another number that gives a value of 0 for the expression. What is this number?

Log in to Amplify Math to complete this lesson online.

Activity 1 Applying the Zero Product Principle

1. Tyler solves the equation $n^2 - 2n = 99$ using the following steps. Analyze his work, and then explain what he did in each step.

	Explanation of each step:
$n^2 - 2n = 99$	**Step 1:** He wrote the original equation.
$n^2 - 2n - 99 = 0$	**Step 2:**
$(n - 11)(n + 9) = 0$	**Step 3:**
$n - 11 = 0$ or $n + 9 = 0$	**Step 4:**
$n = 11$ or $n = -9$	**Step 5:**

2. Solve each equation by rewriting it in factored form and then using the Zero Product Principle.

Equation	Factored form	Solution(s)
$x^2 + 8x + 15 = 0$		
$x^2 - 8x + 12 = 5$		
$x^2 - 10x - 11 = 0$		
$49 - x^2 = 0$		

3. Tyler studies the equation $(x + 4)(x + 5) - 30 = 0$. He concludes that the equation is already in factored form. He sets each factor equal to 30 and determines the solutions are 26 and 25. Do you agree with Tyler? Explain your thinking.

Name: .... Date: .... Period: ....

Activity 2 Revisiting the Zeros of Quadratic Functions

A quadratic equation can have zero, one, or two solutions, which means its graph can have zero, one, or two x-intercepts.

1. Sketch a graph that represents each possible number of x-intercepts for a quadratic function.

Zero x-intercepts	One x-intercept	Two x- intercepts

2. Consider the function $f(x) = x^2 - 2x + 1$.

a Use graphing technology to graph $f(x)$. What do you notice about its x-intercept(s)?

b Solve the equation $x^2 - 2x + 1 = 0$ by writing it in factored form and using the Zero Product Principle. Explain or show your thinking.

3. The function in Problem 2 had only one zero. Write a different quadratic function that has only one zero. Show or explain your thinking.

Activity 3 The Priestess' Garden

Ancient Babylonians and Egyptians often needed to determine the area of a piece of land, but sometimes they did not know its dimensions.

A priestess' square garden has a walkway surrounding it. The total area of the garden and walkway is given by the equation $y = (x + 8)(x + 5)$, where y represents the area in square feet and x represents the side length of the garden in feet. What is the side length of the garden if the total area is 700 ft^2?

1. Write an equation to represent the total area of the garden and walkway.

2. Solve the equation to determine the side length of the square garden.

3. What is the area of the garden? What is the area of the walkway?

STOP

Name: ______ Date: ______ Period: ______

Summary

In today's lesson . . .

You rewrote ***monic quadratic equations*** of the form $ax^2 + bx + c = 0$, where a equals 1, into factored form. In earlier lessons, you noticed that when a quadratic expression is written in factored form, you can efficiently determine values that make the expression equal zero.

Together, these two skills — writing quadratic expressions in factored form and using the Zero Product Principle to determine when a factored expression equals 0 — allow you to solve quadratic equations in other forms.

When a quadratic equation is written as an expression in factored form that equals 0, you can see if the equation has one or two solutions. For example:

- The equation $(x - 3)(x + 1) = 0$ has two solutions, $x = 3$ and $x = -1$.
- The equation $(x + 6)(x + 6) = 0$ has one solution, $x = -6$.

Reflect:

Practice

Name: .... Date: .... Period: ....

1. Use the Zero Product Principle to determine the solution(s) to each equation.

a. $x(x-1)=0$

b. $(5-x)(5+x)=0$

c. $(2x+1)(x+8)=0$

d. $(3x-3)(3x-3)=0$

e. $(7-x)(x+4)=0$

2. Rewrite each equation in factored form, if possible, and solve the equation using the Zero Product Principle.

a. $d^2-7d+6=0$

b. $x^2+18x+81=0$

c. $u^2+7u-60=0$

d. $x^2+0.2x+0.01=0$

3. Elena is solving the quadratic equation $x^2-3x-18=0$. Her work is shown. Do you agree or disagree with her work? If you disagree, explain the error and correct it. Otherwise, check Elena's solutions by substituting them into the original equation and showing that the equation is true.

Elena's work:

$x^2-3x-18=0$

$(x-3)(x+6)=0$

$x-3=0$ or $x+6=0$

$x=3$ or $x=-6$

4. Jada is solving the quadratic equation $p^2 - 5p = 0$. Her work is shown. Jada says her solution is correct because substituting 5 for p into the original expression gives $p^2 - 5p = 5^2 - 5(5) = 25 - 25 = 0$. Explain Jada's mistake and determine the correct solutions.

Jada's work:

$$p^2 - 5p = 0$$
$$p(p-5) = 0$$
$$p - 5 = 0$$
$$p = 5$$

5. Write each expression in factored form. If it is not possible, write *not possible*.

a. $x^2 - 144$

b. $x^2 + 16$

c. $25 - x^2$

d. $b^2 - a^2$

e. $100 + y^2$

6. Expand the expression to write an equivalent expression in standard form.

$(2x + 4)(x + 2)$

Unit 6 | Lesson 10

Writing Non-Monic Quadratic Expressions in Factored Form

Let's write non-monic quadratic expressions in factored form.

Warm-up Which One Doesn't Belong?

Study each expression. Which expression does not belong? Explain your thinking.

A. $(x+4)(x-3)$

B. $3x^2-8x+5$

C. x^2-25

D. x^2-3x+3

Log in to Amplify Math to complete this lesson online.

Name: ______________________ Date: __________ Period: __________

Activity 1 Yes, You Can!

You previously factored monic quadratic expressions and quadratics that were a difference of squares. Refer to these quadratic expressions.

$9x^2 + 21x + 10$	$3x^2 - 8x + 5$

1. What do you notice about these expressions compared to ones you have previously factored?

2. Think of all the strategies you have previously used to factor quadratic expressions. Use any of these strategies to help you to factor these expressions.

   a. $9x^2 + 21x + 10$

   b. $3x^2 - 8x + 5$

Activity 2 There's a Strategy for That

Jada and Clare use different strategies to factor the quadratic expressions from Activity 1.

$9x^2 + 21x + 10$	$3x^2 - 8x + 5$

1. Jada studies the expression $9x^2 + 21x + 10$ and notices that $9x^2$ is a square term (both the variable and the coefficient are squares).

a Rewrite $9x^2$ to show that it is the square of another expression.

b Jada notices that the terms $9x^2$ and $21x$ share a common factor. What is the greatest common factor of $9x^2$ and $21x$?

c Jada is excited! She has made a discovery. Study her work, then explain Jada's discovery.

Jada's work:

$$\begin{aligned} &9x^2 + 21x + 10 \\ &= (3x)^2 + 7(3x) + 10 \\ &= N^2 + 7N + 10 \\ &= (N + 2)(N + 5) \\ &= (3x + 2)(3x + 5) \end{aligned}$$

d Use Jada's strategy to write these quadratic expressions in factored form.

$4x^2 + 28x + 45$ $\qquad$ $25x^2 - 35x + 6$

2. Clare notices that Jada's strategy does not work for the expression $3x^2 - 8x + 5$.

a Why might Jada's strategy not work for this expression? Explain your thinking.

b Clare notices that she can make the coefficient of the squared term $3x^2$ a perfect square by multiplying the entire expression $3x^2 - 8x + 5$ by 3. What is the resulting expression after multiplying by 3?

c Clare decides that she can now use Jada's strategy to factor the expression and writes the factored expression $(3x - 3)(3x - 5)$. She then uses the Distributive Property to expand her expression to check her work. What is the product of Clare's expression? Is it equivalent to the original expression, $3x^2 - 8x + 5$?

Name: ________ Date: ________ Period: ________

Activity 2 There's a Strategy for That (continued)

d Clare discovers her error and corrects her work. Study Clare's work. Explain the strategy she used.

Clare's work:

$$\left(\frac{1}{3}\right)(3x - 3)(3x - 5)$$
$$= (x - 1)(3x - 5)$$

e Use Clare's strategy to write these expressions in factored form.

$3x^2 + 16x + 5$ $\qquad$ $10x^2 - 41x + 4$

3. Complete the table. Determine whether Jada's or Clare's strategy would be more useful for factoring each expression.

Factored form	Standard form	More useful strategy?
$(3x + 1)(x + 4)$		
	$6x^2 + 19x + 10$	
$(3x + 2)(x + 2)$		
$(3x + 4)(x + 1)$		
	$4x^2 - 14x - 30$	

Are you ready for more?

Three quadratic equations are shown, each with two solutions. Use the Zero Product Principle to determine both solutions to each equation.

$x^2 = 6x$ $\qquad$ $x(x + 4) = x + 4$ $\qquad$ $2x(x - 1) + 3x = 3$

Activity 3 Timing a Drop of Water

An engineer designs a fountain that shoots drops of water upward from a nozzle that is 3 m above the ground, at a vertical velocity of 9 m per second. The height h, in meters, of a drop of water t seconds after it is shot from the nozzle is defined by the function $h(t) = -5t^2 + 9t + 3$.

When will the drop of water hit the ground?

1. Write an equation that can be used to solve this problem.

2. Try to solve the equation by writing the expression in factored form and using the Zero Product Principle. What do you notice?

3. Use graphing technology to solve the equation by graphing. Explain how you determined the solution.

STOP

Name: ______________________ Date: __________ Period: __________

Summary

In today's lesson . . .

You observed that writing ***non-monic quadratic equations*** of the form $ax^2 + bx + c = 0$ in factored form is not always the most efficient way to determine its solutions. Determining the factors of non-monic quadratic expressions is often challenging. And sometimes the solutions are not even rational numbers.

It turns out that writing quadratic expressions in factored form and using the Zero Product Principle is a limited tool that only works for some quadratic equations. In the coming lessons, you will learn strategies to solve *any* quadratic equation.

Reflect:

Practice

Name: .... Date: .... Period: ....

1. To write the expression $11x^2 + 17x - 10$ in factored form, Diego first lists some of the factor pairs of the constant term -10. Shown are the possible factorizations. Which expression on Diego's list is equivalent to the expression $11x^2 + 17x - 10$? Explain your thinking.

$(11x+5)(x-2)$	$(x+5)(11x-2)$
$(11x+2)(x-5)$	$(x+2)(11x-5)$
$(11x+10)(x-1)$	$(x+10)(11x-1)$
$(11x+1)(x-10)$	$(x+1)(11x-10)$

2. To rewrite the expression $4x^2 - 12x - 7$ in factored form, Jada listed some factor pairs of the term $4x^2$. Write the remaining factor pairs, then determine which expression is equivalent to $4x^2 - 12x - 7$.

$(2x$ ....$)(2x$ ....$)$ $(4x$ ....$)(x$ ....$)$

$(2x$ ....$)(2x$ ....$)$ $(4x$ ....$)(x$ ....$)$

$(4x$ ....$)(x$ ....$)$ $(4x$ ....$)(x$ ....$)$

3. Han solves the equation $5x^2 + 13x - 6 = 0$. His work is shown. Describe Han's mistake. Then determine the correct solutions to the equation.

Han's work:

$5x^2 + 13x - 6 = 0$

$(5x - 2)(x + 3) = 0$

$x = 2$ or $x = -3$

Name: ______ Date: ______ Period: ______

Practice

4. A picture measures 10 in. wide and 15 in. long. The area A of the picture, including a frame with a thickness of x in. all the way around, can be modeled by the function $A(x) = (2x + 10)(2x + 15)$.

   a. Use function notation to write a statement that represents the following statement: *The area of the picture, including a frame with a thickness of 2 in., is 266 in^2.*

   b. What is the total area if the picture has a frame with a thickness of 4 in.?

5. To solve the equation $0 = 4x^2 - 28x + 39$, Elena uses graphing technology to graph the function $f(x) = 4x^2 - 28x + 39$. She determines that the graph intersects the x-axis at the points $(1.919, 0)$ and $(5.081, 0)$.

   a. What is the name for the points at which the graph of a function crosses the x-axis?

   b. Use a calculator to compute $f(1.919)$ and $f(5.081)$.

   c. Explain why 1.919 and 5.081 are *approximate* solutions to the equation $0 = 4x^2 - 28x + 39$, and not *exact* solutions.

6. Solve each equation.

   a. $x^2 = 16$

   b. $x^2 = 49$

   c. $x^2 = 100$

My Notes:

SUB-UNIT

3 Completing the Square

Narrative Connections

How many ways can you crack an egg?

When Lemuel Gulliver washed onto the island of Lilliput, he found a nation of six-inch tall people at war with their neighbors. The source of their conflict? They couldn't agree which was the right end of the egg to crack.

So, what can we learn from Jonathan Swift's smirking satire, *Gulliver's Travels*?

Like Gulliver, we know there's more than one way to crack an egg. And so too, for our math problems.

At its core, the history of math is the history of how people solve problems. All developing civilizations faced the same kinds of problems, like how to grow sufficient numbers of crops, build storehouses, and collect taxes. And to solve their problems efficiently, they had to develop their own mathematical methods.

You can see this in the variety of ways different societies approached the problem of quadratic equations.

Building on their discovery of zero and negative numbers, the Indians and Persians in the 7th century tackled these problems *algebraically* — laying out specific procedures for multiplying and taking the square roots of different terms.

But more than a thousand years earlier, ancient Babylonian mathematicians, along with ancient Chinese mathematicians, approached the problem *geometrically*, making use of everyone's favorite quadrilateral — the square!

Despite being scattered across the Asian continent and separated by millennia, these societies drew on their individual creativity, history, and learning to arrive at the same answer, through solutions that were uniquely their own.

Unit 6 | Lesson 11

Square Expressions

Let's examine how perfect squares can help us solve some quadratic equations more efficiently.

Warm-up What is a ?

For each equation, determine an expression for a so that the equation is true for all values of x.

1. $x^2 = a^2$

2. $(3x)^2 = a^2$

3. $25x^2 = a^2$

4. $a^2 = (x + 1)^2$

Compare and Connect: What do you notice about the structure of these equations and how you determined what a could represent? Be prepared to share your thoughts with the class.

Log in to Amplify Math to complete this lesson online.

Name: ________________ Date: ________ Period: ________

Activity 1 Squares in Different Forms

1. Write an equivalent expression in standard form for each expression.

a $(3x)^2$

b $7x \cdot 7x$

c $(x+4)(x+4)$

d $(x+1)^2$

e $(x-7)^2$

f $(x+n)^2$

2. Each of the following is considered a *square expression*. Why do you think that is? Explain or show your thinking.

$x^2 + 6x + 9$	$x^2 - 16x + 64$	$x^2 + \frac{1}{3}x + \frac{1}{36}$

3. Write each square expression in factored form.

a $x^2 + 10x + 25$

b $x^2 - 16x + 64$

c $x^2 - \frac{1}{2}x + \frac{1}{16}$

d $49x^2 + 84x + 36$

Are you ready for more?

Write each expression using $(mx + p)^2$ form.

1. $x^4 - 30x^2 + 225$
2. $x + 14\sqrt{x} + 49$
3. $5^{2x} + 6 \cdot 5^x + 9$

Activity 2 Two Strategies

Han and Lin solved the same equation using different strategies. Their work is shown. What do you notice? What do you wonder?

Han's strategy:	**Lin's strategy:**
$(x-6)^2 = 25$ $(x-6)(x-6) = 25$ $x^2 - 12x + 36 = 25$ $x^2 - 12x + 11 = 0$ $(x-11)(x-1) = 0$ $x - 11 = 0$ or $x - 1 = 0$ $x = 11$ or $x = 1$	$(x-6)^2 = 25$ $x - 6 = 5$ or $x - 6 = -5$ $x = 11$ or $x = 1$

1. I notice . . .

2. I wonder . . .

3. Four equations are shown. Work with a partner to solve each equation. One partner should use Han's strategy and the other partner should use Lin's strategy. You should arrive at the same solutions. If not, work together to resolve any disagreements.

   a. $(y-5)^2 = 49$

   b. $(2x+2)^2 = 16$

   c. $\left(x + \frac{1}{3}\right)^2 = \frac{4}{9}$

   d. $(\nu - 0.1)^2 = 0.36$

STOP

Name: ________ Date: ________ Period: ________

Summary

In today's lesson . . .

You observed . . .

- *square numbers* like 9, which is 3^2 or $(-3)^2$,
- *square terms* like $9x^2$, which is $(3x)^2$ or $(-3x)^2$, and
- quadratics that are ***square expressions***, the product of a linear expression (or any expression, really) and itself.

Quadratics that are square expressions are written in standard form as $ax^2 + 2abx + b^2$, and in factored form as $(ax + b)^2$.

Whenever you see a square expression in a quadratic equation, you can factor the expression to help you solve the equation. For example:

$x^2 + 6x + 9 = 16$	The square expression is $x^2 + 6x + 9$.
$(x + 3)^2 = 16$	Factor the square expression.
$x + 3 = \pm 4$	Take the square root of each side.
$x = 1$ or $x = -7$	Solve the two equations.

As you will see, square expressions can be very helpful for solving quadratic equations.

Reflect:

Practice

Name: ______ Date: ______ Period: ______

1. Select *all* the expressions that are square expressions.

A. $(x+5)(x+5)$

B. $(-9+c)(c-9)$

C. $(y-10)(10-y)$

D. $(a+3)(3+a)$

E. $(2x-1)(2x+1)$

F. $(4-3x)(3-4x)$

G. $(a+b)(b+a)$

H. $16x^2-48x+36$

2. Solve each equation. Explain or show your thinking.

a. $(x-1)^2=4$

b. $(x+5)^2=81$

c. $(x-2)^2=0$

d. $(x-11)^2=121$

e. $(x-7)^2=\frac{64}{49}$

f. $(x-4)^2=36$

3. The equations $y=(x+2)(x+3)$ and $y=x^2+5x+6$ are equivalent.

a. Which equation would you use to determine the x-intercepts? Explain your thinking.

b. Which equation would you use to determine the y-intercept? Explain your thinking.

Name: .... Date: .... Period: ....

4. Select *all* the equations with a positive y-intercept.

A. $y = x^2 + 3x - 2$

B. $y = x^2 - 10x$

C. $y = (x - 1)^2$

D. $y = 5x^2 - 3x - 5$

E. $y = (x + 1)(x + 2)$

F. $y = x^2 + 2x + 3$

5. Which of the following is a square number?

A. 14

B. 24

C. 36

D. 56

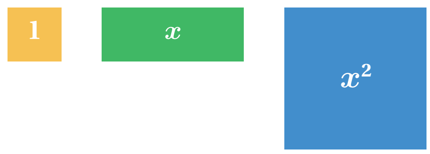
6. What value would need to go in the blank space to make $x^2 + 4x +$ .... equivalent to $(x + 2)^2$?

Completing the Square

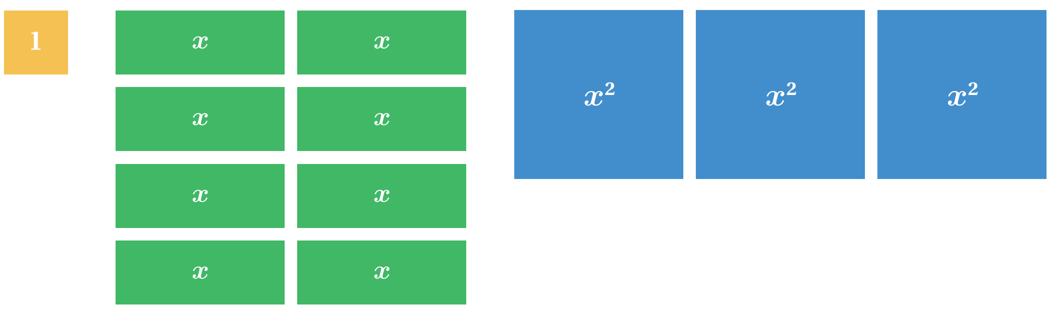
Let's learn a new strategy for solving quadratic equations.

Warm-up Wholesome Squares

The algebra tile models represent 1, x, and x^2.



Create as many different square quadratic expressions as you can using the algebra tiles provided. Explain your thinking.



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Name: ________ Date: ________ Period: ________

Activity 1 Is It Square?

1. For each of the following, determine whether each expression is a square expression. Write *Yes* or *No*. Be prepared to show or explain your thinking.

   a. $(x + 5)(5 + x)$

   b. $(x + 5)(x - 5)$

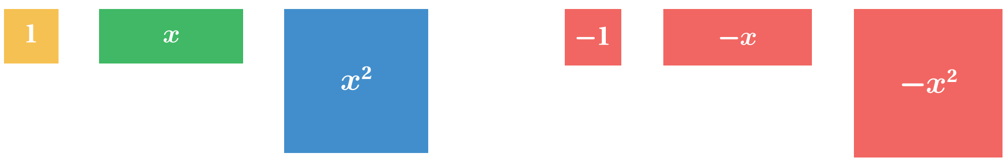
   c. $(x - 3)^2$

   d. $x - 3^2$

   e. $x^2 + 8x + 16$

   f. $x^2 + 10x + 20$

2. Use algebra tiles to verify whether each expression from Problem 1 is a square expression. Draw a sketch of your algebra tiles in the space provided.



   a. $(x + 5)(5 + x)$

   b. $(x + 5)(x - 5)$

   c. $(x - 3)^2$

   d. $x - 3^2$

   e. $x^2 + 8x + 16$

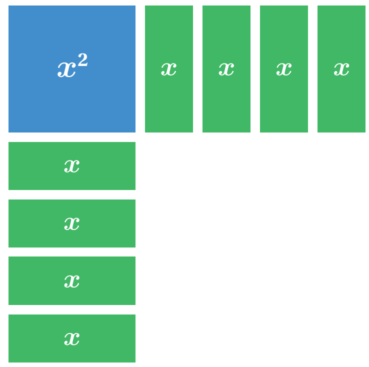
   f. $x^2 + 10x + 20$

Activity 2 Building Complete Squares

In each problem, a set of algebra tiles is shown. For each problem, complete these tasks:

- Sketch the missing algebra tiles needed to complete each square.
- Describe how the algebra tiles relate to the missing values.
- Complete the equation.

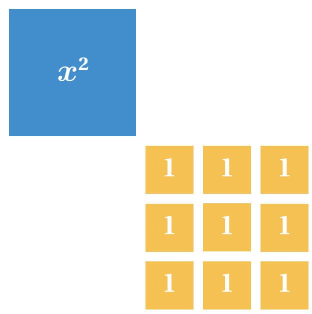
1. Description:



Complete the equation:

$x^2 + 8x + \underline{\qquad} = (x + \underline{\qquad})^2$

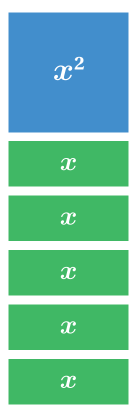
2. Description:



Complete the equation:

$x^2 + \underline{\qquad}x + 9 = (x + \underline{\qquad})^2$

3. Description:



Complete the equation:

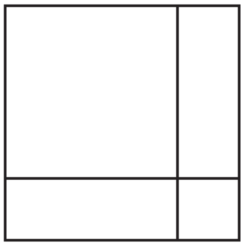
$x^2 + \underline{\qquad}x + \underline{\qquad} = (x + 5)^2$

Name: ______ Date: ______ Period: ______

Activity 3 Algebraically Building Complete Squares

1. Consider the quadratic expression $x^2 + 6x + 9$.

a Complete the area diagram and relate each term of the expression to its corresponding rectangle.

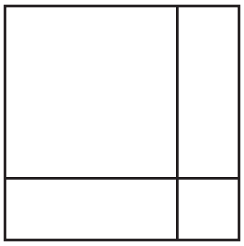


b Write the expression in factored form.

c How does the constant term in factored form relate to the linear term in standard form?

2. Consider the incomplete quadratic expression $x^2 - 20x + \_\_\_$.

a Complete the area diagram to determine the value of the missing term.

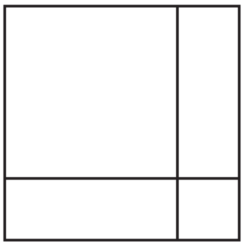


b Write the expression in factored form.

c How does the constant term in standard form relate to the constant term in factored form?

3. Consider the incomplete quadratic expression $x^2 + 7x + \_\_\_$.

a Complete the area diagram to determine the value of the missing term.



b Write the expression in factored form.

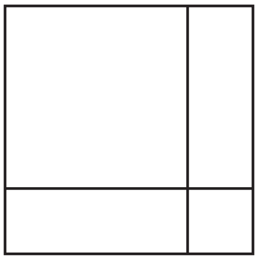
c How does the coefficient of the linear term in standard form relate to:

- The constant term in factored form?
- The constant term in standard form?

Activity 3 Algebraically Building Complete Squares (continued)

4. Consider the incomplete quadratic expression $x^2 + bx +$ ___.

a Create an area diagram to determine the value of the missing term.

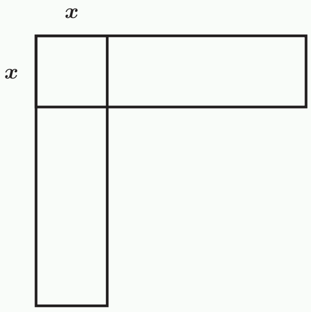


b Write the expression in factored form.

c Complete the equation $x^2 + bx +$ .... $= (x +$ ....$)^2$.

Are you ready for more?

Consider the figure, which contains one square and two congruent rectangles. The total area of the figure is $x^2 + 35x$ square units.



1. Determine the length of the longer unlabeled side of each of the two rectangles.

2. If you draw lines and connect them to form a larger square, what will be the area of the entire figure?

3. How is the process of determining the area of the entire figure similar to the process of building complete squares for expressions such as $x^2 + bx$?

STOP

Reflect: How were you able to manage your stress when the activity became challenging?

Name: ________________ Date: ________ Period: ________

Summary

In today's lesson . . .

You added or subtracted specific constant values from quadratic expressions, turning them into square expressions. For example, if you have the expression $x^2 + bx$, you can add $\left(\frac{b}{2}\right)^2$. Then, your new expression is $x^2 + bx + \left(\frac{b}{2}\right)^2$, a square expression that is equivalent to $\left(x + \frac{b}{2}\right)^2$.

Adding (or subtracting) a constant term to a quadratic expression to make it a square expression is called ***completing the square***. In the next lesson, you will complete the square as a strategy for solving quadratic equations.

Reflect:

Practice

Name: ________ Date: ________ Period: ________

1. For each expression, determine the value that, when added to the expression, makes it a square expression. Write the square expression in both standard form and factored form.

Expression	Square expression in standard form	Square expression in factored form
$x^2 - 6x$		
$x^2 + 2x$		
$x^2 + 14x$		
$x^2 - 4x$		
$x^2 + 24x$		

2. Each of the following expressions written in standard form is a square expression that is missing either a coefficient or a constant term. Determine the missing value. Then match each expression in standard form with an equivalent square expression in factored form.

Standard Form	Factored Form
a. $x^2 + 8x +$ ______	______ $(x - 10)^2$
b. $x^2 +$ ______ $x + 25$	______ $(x - 7)^2$
c. $x^2 - 14x +$ ______	______ $(x + 4)^2$
d. $x^2 -$ ______ $x + 100$	______ $(x + 5)^2$

3. Mai is changing the expression $x^2 + 12x$ so that it will be a square expression. Her work is shown. Jada studies Mai's work, but does not understand exactly what Mai did to change the expression. Complete Mai's missing step to help Jada see how Mai changed the expression.

Mai's work:
$x^2 + 12x$
$(x + 6)^2$

Name: .... Date: .... Period: ....

Practice

4. Write each quadratic expression in standard form.

a) $(x+3)(x-3)$

b) $(7+x)(x-7)$

c) $(2x-5)(2x+5)$

d) $\left(x+\frac{1}{8}\right)\left(x-\frac{1}{8}\right)$

5. To determine the value of the expression $203 \bullet 197$ without using a calculator, Priya writes $(200+3)(200-3)$, and calculates the product $39{,}991$. Explain why writing the two factors as a sum and a difference is a useful strategy here.

6. Solve each equation for *all* values of x that make the equation true.

a) $(x+1)^2 = 9$

b) $\left(x-\frac{1}{3}\right)^2 = 4$

c) $(x+2)^2 = 8+1$

d) $(x-0.5)^2 + 0.6 = 1.6$

e) $(x-3)(x-3) = 49$

Unit 6 | Lesson 13

Solving Quadratic Equations by Completing the Square

Let's see if completing the square can help us solve equations.

Warm-up Math Talk

What strategy would you use to evaluate or solve each equation? Use your strategy to determine the solution. Explain your thinking.

1. $x + x = \frac{1}{4}$

Solution:

Strategy:

2. $\left(\frac{3}{2}\right)^2 = x$

Solution:

Strategy:

3. $\frac{3}{5} + x = \frac{9}{5}$

Solution:

Strategy:

4. $\frac{1}{12} + x = \frac{1}{4}$

Solution:

Strategy:

Log in to Amplify Math to complete this lesson online.

Name: .... Date: .... Period: ....

Activity 1 Solving by Completing the Square

One way to solve quadratic equations is to complete the square. Diego and Mai used two different strategies to solve the same quadratic equation, $x^2 + 10x + 9 = 0$. Study their strategies. What do you notice? What do you wonder?

Diego's strategy:	Mai's strategy:
$x^2 + 10x + 9 = 0$	$x^2 + 10x + 9 = 0$
$x^2 + 10x = -9$	$x^2 + 10x + 9 + 16 = 16$
$x^2 + 10x + 25 = -9 + 25$	$x^2 + 10x + 25 = 16$
$x^2 + 10x + 25 = 16$	$(x + 5)^2 = 16$
$(x + 5)^2 = 16$	$x + 5 = -4$ or $x + 5 = 4$
$x + 5 = -4$ or $x + 5 = 4$	$x = -9$ or $x = -1$
$x = -9$ or $x = -1$	

1. I notice . . .

2. I wonder . . .

Activity 1 Solving by Completing the Square (continued)

3. Use a separate sheet of paper to solve each equation using Diego's or Mai's strategy of completing the square, as indicated. Record your solutions in the table.

Use Diego's strategy:	Use Mai's strategy:
$x^2 + 6x + 8 = 0$	$x^2 + 12x = 13$
$0 = x^2 - 10x + 21$	$x^2 - 2x + 3 = 83$
$x^2 - 8x + 7 = 0$	$x^2 + 40 = 14x$

4. Whose strategy do you prefer? Explain your thinking.

Are you ready for more?

1. Show that the equation $x^2 + 10x + 9 = 0$ is equivalent to $(x + 3)^2 + 4x = 0$.

2. Write an equation that is equivalent to $x^2 + 9x + 16 = 0$, and which also contains the expression $(x + 4)^2$.

3. Do these equivalent equations help you determine the solutions to the original equations? Explain your thinking.

Name: ______ Date: ______ Period: ______

Activity 2 Solving More Interesting Equations

Solve each equation by completing the square. Show your thinking.

1. $(x-3)(x+1)=5$

2. $x^2+\frac{1}{2}x=\frac{3}{16}$

3. $x^2+3x+\frac{8}{4}=0$

4. $(7-x)(3-x)+3=0$

5. $x^2+1.6x+0.63=0$

6. $x^2+5x=9.75$

Historical Moment

Solving Equations Without Symbols

What is the difference between algebra and arithmetic? Many people think the difference is all those fancy symbols — but algebraic thinking has been around for centuries to help make sense of everyday problems, long before these symbols were invented.

Consider the solution to the quadratic equation: $x^2+21=10x$:

"Halve the number of the roots. It is 5. Multiply this by itself and the product is 25. Subtract from this the 21 added to the square term and the remainder is 4. Extract its square root, 2, and subtract this from half the number of roots, 5. There remains 3. This is the root you wanted, whose square is 9. Alternatively, you may add the square root to half the number of roots and the sum is 7. This is then the root you wanted and the square is 49."

Complete the square of $x^2+21=10x$. Then explain how people from centuries ago would describe completing the square.

Activity 3 Find and Fix

For each equation, complete these tasks:

- **Solve the equation by completing the square.**
- **Review the worked solution. Each worked solution contains at least one error.**
- **Find and describe any errors in the worked solution.**

	Worked solutions (with errors)	Describe the error(s):
1. $x^2 + 14x = -24$ **Correct solution(s):**	$x^2 + 14x = -24$ $x^2 + 14x + 28 = 4$ $(x + 7)^2 = 4$ $x + 7 = -2$ or $x + 7 = 2$ $x = -9$ or $x = -5$	
2. $x^2 - 10x + 16 = 0$ **Correct solution(s):**	$x^2 - 10x + 16 = 0$ $x^2 - 10x + 25 = 9$ $(x - 5)^2 = 9$ $x - 5 = -9$ or $x - 5 = 9$ $x = -4$ or $x = 14$	
3. $x^2 + 2.4x = -0.8$ **Correct solution(s):**	$x^2 + 2.4x = -0.8$ $x^2 + 2.4x + 1.44 = 0.64$ $(x + 1.2)^2 = 0.64$ $x + 1.2 = 0.8$ $x = -0.4$	
4. $x^2 - \frac{6}{5}x + \frac{1}{5} = 0$ **Correct solution(s):**	$x^2 - \frac{6}{5}x + \frac{1}{5} = 0$ $x^2 - \frac{6}{5}x + \frac{9}{25} = \frac{9}{25}$ $\left(x - \frac{3}{5}\right)^2 = \frac{9}{25}$ $x - \frac{3}{5} = -\frac{3}{5}$ or $x - \frac{3}{5} = \frac{3}{5}$ $x = 0$ or $x = \frac{6}{5}$	

STOP

Name: ______ Date: ______ Period: ______

Summary

In today's lesson . . .

You saw that completing the square can be a useful strategy for solving quadratic equations. Here are two strategies for solving by completing the square, along with an example of each.

- In Strategy 1, the constant term is first moved to the other side of the equation.
- In Strategy 2, a value is added or subtracted to make the constant term the required square number.

Solve the quadratic equation $x^2 + 4x - 5 = 0$ by completing the square.

Strategy 1	Strategy 2
First move the constant term to the other side of the equation. $x^2 + 4x - 5 = 0$ $x^2 + 4x = 5$ $x^2 + 4x + 4 = 5 + 4$ $x^2 + 4x + 4 = 9$ $(x+2)^2 = 9$ $x + 2 = -3$ or $x + 2 = 3$ $x = -5$ or $x = 1$	Add or subtract to the constant term to get the required square number. $x^2 + 4x - 5 = 0$ $x^2 + 4x - 5 + 9 = 9$ $x^2 + 4x + 4 = 9$ $(x+2)^2 = 9$ $x + 2 = -3$ or $x + 2 = 3$ $x = -5$ or $x = 1$

Reflect:

Practice

Name: .... Date: .... Period: ....

1. Noah wants to solve the quadratic equation $x^2 + 8x + 15 = 0$ by completing the square. His work is shown. He says that there is one solution, $x = -4$. Do you agree with Noah? Why or why not?

Noah's work:

$$x^2 + 8x + 15 = 0$$
$$x^2 + 8x + 15 + 1 = 0$$
$$x^2 + 4x + 16 = 0$$
$$(x + 4)^2 = 0$$

2. Solve each equation by completing the square.

a $x^2 - 6x + 5 = 12$

b $x^2 - 2x = 8$

c $11 = x^2 + 4x - 1$

d $x^2 - 18x + 60 = -21$

3. Three quadratic equations and their solutions are shown. Explain or show how to solve each equation by completing the square.

a $x^2 + 20x + 50 = 14$
Solutions: $x = -18$
and $x = -2$

b $x^2 + 1.6x = 0.36$
Solutions: $x = -1.8$
and $x = 0.2$

c $x^2 - 5x = \frac{11}{4}$
Solutions: $x = -\frac{1}{2}$
and $x = \frac{11}{2}$

Name: .... Date: .... Period: ....

Practice

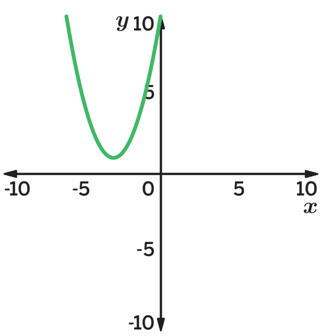
4. Match each quadratic expression given in factored form with an equivalent expression in standard form. One expression in standard form does not have an equivalent factored form.

Factored Form	Standard Form
a $(2+x)(2-x)$	.... $x^2 - 4$
b $(x+9)(x-9)$	.... $81 - x^2$
c $(2+x)(x-2)$	.... $x^2 - 81$
	.... $4 - x^2$

5. Three equations and three graphs are shown. Write each equation underneath its corresponding graph. Explain your thinking.

$y = x^2 + 4x + 3$ $\qquad$ $y = (x+2)(x+3)$ $\qquad$ $y = (x+3)^2 + 1$

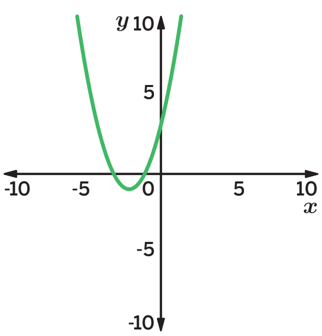
Graph 1



Equation:

Explanation:

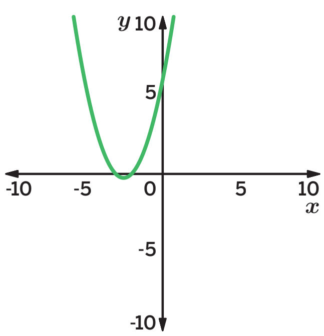
Graph 2



Equation:

Explanation:

Graph 3



Equation:

Explanation:

Unit 6 | Lesson 14

Writing Quadratic Expressions in Vertex Form

Let's find other uses for completing the square.

Warm-up Three Expressions, One Function

The following are three equivalent functions, each in a different form.

$f(x) = x^2 + 6x + 8$ $\qquad g(x) = (x + 2)(x + 4)$ $\qquad h(x) = (x + 3)^2 - 1$

Select a function to determine the following features of the function's graph.

1. The vertex

2. The x-intercepts

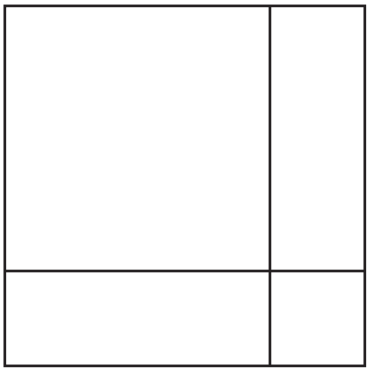
3. The y-intercept

Log in to Amplify Math to complete this lesson online.

Name: ______ Date: ______ Period: ______

Activity 1 Adding and Subtracting

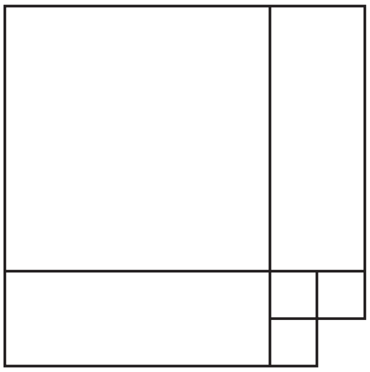
1. Consider the quadratic expression $x^2 + 4x + 4$.



a Label the area diagram and relate each term in the expression to its corresponding rectangle(s).

b Write the expression in factored form.

2. Consider the quadratic expression $x^2 + 4x + 3$.



a Label the area diagram and relate each term of the expression to its corresponding rectangle(s).

b Based on your diagram, how many unit squares are needed to complete the square?

c Adding a value to complete the square would *change* the expression. Rewrite an equivalent expression by adding and subtracting the same value.

$x^2 + 4x + 3 +$ ______ $-$ ______

d Rewrite the expression $x^2 + 4x + 3$ in vertex form by completing these steps.

$x^2 + 4x + 3$

$x^2 + 4x + 3 +$ ______ $-$ ______

$x^2 + 4x +$ ______ $-$ ______

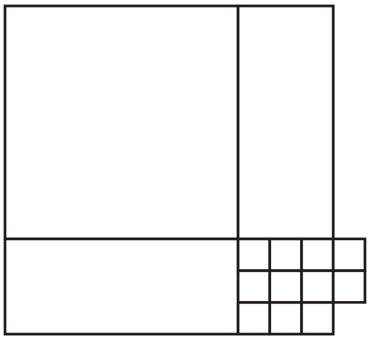
$(x +$ ______$)^2 -$ ______

e Describe the relationship between the vertex form and the area diagram.

Activity 1 Adding and Subtracting (continued)

3. Consider the expression, $x^2 + 6x + 11$.

a Label the area diagram and relate each term of the expression to its corresponding rectangle(s).



b Based on your diagram, how many unit squares are preventing the complete square?

c Subtracting a value to complete the square would *change* the expression. Rewrite an equivalent expression by subtracting and adding the same value.

$x^2 + 6x + 11 -$ .... $+$ ....

d Rewrite the expression $x^2 + 6x + 11$ in vertex form. Show your thinking.

e Explain the process you used to rewrite the expression $x^2 + 6x + 11$ in vertex form.

4. Consider the expression $x^2 - 6x + 7$.

a Create an area diagram for the expression.

b Rewrite the expression in vertex form. Show your thinking.

Name: ________________ Date: ________ Period: ________

Activity 2 Decomposing c

Recall that the vertex form of a quadratic expression is $(x-h)^2+k$ and that the standard form of a quadratic expression is ax^2+bx+c.

1. Several quadratic expressions in vertex form are shown. Identify the values of h and k. Then rewrite each quadratic expression in standard form and identify the value of c.

Vertex form	Value of h	Value of k	Standard form	Value of c
$(x+5)^2+1$				
$(x-6)^2+4$				
$(x+1)^2-2$				
$(x-3)^2-7$				

2. Study the relationship between the values of h, k, and c in each pair of equivalent expressions. What do you notice?

3. Use what you noticed in Problem 2 to write the standard form for each expression *without* expanding the vertex form.

Vertex form	Standard form
$(x+5)^2+3$	
$(x-6)^2+7$	
$(x+1)^2-6$	
$(x-3)^2-1$	

Activity 2 Decomposing c (continued)

4. Consider $x^2 + 10x + 32$, a quadratic expression in standard form.

a Study the first two terms. What constant would need to be added to complete the square for the expression $x^2 + 10x$?

b What constant was added to $x^2 + 10x$ instead of the value you determined in part a? To complete the square, how could you decompose this constant into a sum of two numbers?

c Write the expression $x^2 + 10x + 32$ in vertex form by completing the square.

d Use the relationship you discovered in Problem 2 to verify the vertex form you wrote in part c is equivalent to $x^2 + 10x + 32$. Explain your thinking.

e Expand your expression in vertex form to confirm it is equivalent to $x^2 + 10x + 32$.

5. Rewrite the following expressions in vertex form by first decomposing the value of c.

a $x^2 - 2x + 9$

b $x^2 + 10x + 9$

c $x^2 + 4x + 3$

Name: ______________________ Date: __________ Period: __________

Activity 3 Info Gap: Features of Functions

You will receive either a problem card or a data card. Do not show or read your card to your partner.

If you are given the *data* card:	If you are given the *problem* card:
1. Silently read the information on your card.	**1.** Silently read your card and think about what information you need to solve the problem.
2. Ask your partner, "What specific information do you need?" and wait for your partner to ask for information. Only give information that is on your card.	**2.** Ask your partner for the specific information that you need.
3. Before telling your partner the information, ask, "Why do you need to know (that piece of information)?"	**3.** Explain to your partner how you are using the information to solve the problem.
4. Read the problem card, and solve the problem independently.	**4.** When you have enough information, share the problem card with your partner, and solve the problem independently.
5. Share the data card, and discuss your thinking.	**5.** Read the data card, and discuss your thinking.

STOP

Summary

In today's lesson . . .

You converted quadratic expressions, such as $x^2 - 2x + 9$, from standard form to vertex form using different strategies:

Decompose.

$x^2 - 2x + \mathbf{9}$
$x^2 - 2x + \mathbf{1 + 8}$
$(x - 1)^2 + 8$

Subtract, then add.

$x^2 - 2x + 9$
$x^2 - 2x + 9 \mathbf{- 8 + 8}$
$x^2 - 2x + \mathbf{1 + 8}$
$(x - 1)^2 + 8$

Add, then subtract.

$x^2 - 2x + 9$
$x^2 - 2x \mathbf{+ 1 - 1} + 9$
$x^2 - 2x + \mathbf{1 + 8}$
$(x - 1)^2 + 8$

Each of these strategies completes the square.

Quadratic functions can have different equivalent forms. While each form defines the same quadratic function, all forms reveal key information about the function's graph.

- In the standard form of a quadratic, $ax^2 + bx + c$, the value of c indicates the y-intercept.
- In the factored form of a monic quadratic, $(x + p)(x + q)$, the values of p and q indicate the x-intercepts of the function $(-p, 0)$ and $(-q, 0)$.
- In the vertex form of a quadratic, $a(x - h)^2 + k$, the values of h and k indicate the vertex of the graph, which has the coordinates (h, k).

Reflect:

Name: .... Date: .... Period: ....

1. The following quadratic functions are equivalent. Select *all* the statements that are true about the graphs of the functions.

$f(x) = (x + 5)(x + 3)$ $\quad$ $g(x) = x^2 + 8x + 15$ $\quad$ $h(x) = (x + 4)^2 - 1$

A. The vertex is located at $(4, -1)$.

B. The vertex is located at $(-4, -1)$.

C. The y-intercept is 15.

D. The x-intercept is located at $(0, 15)$.

E. The y-intercept is -15.

F. The x-intercepts are located at $(0, 5)$ and $(0, 3)$.

G. The x-intercepts are located at $(-5, 0)$ and $(-3, 0)$.

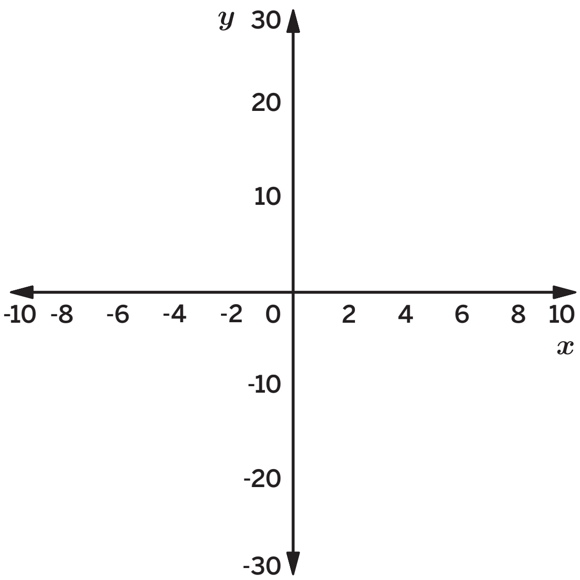
2. Consider the quadratic function $g(x) = x^2 + 2x - 24$.

a Determine the y-intercept.

b Determine the x-intercepts of the function.

c Determine the vertex of the function.

d Sketch a graph of the function.



Practice

Name: .... Date: .... Period: ....

3. Consider the steps taken to rewrite the following expression in vertex form.

$$x^2 - 7x + 6$$
$$x^2 - 7x + \left(-\frac{7}{2}\right)^2 + 6 - \left(-\frac{7}{2}\right)^2$$
$$\left(x - \frac{7}{2}\right)^2 + 6 - \frac{49}{4}$$
$$\left(x - \frac{7}{2}\right)^2 + \frac{24}{4} - \frac{49}{4}$$
$$\left(x - \frac{7}{2}\right)^2 - \frac{25}{4}$$

a. Study the second line. Where did the value $\left(-\frac{7}{2}\right)^2$ come from? Why was it both added and subtracted?

b. Explain what happened in the third line.

4. How is the graph of the equation $y = (x - 1)^2 + 4$ related to the graph of the equation $y = x^2$?

A. The graph of $y = (x - 1)^2 + 4$ is the same as the graph $y = x^2$ but is shifted 1 unit to the right and 4 units up.

B. The graph of $y = (x - 1)^2 + 4$ is the same as the graph $y = x^2$ but is shifted 1 unit to the left and 4 units up.

C. The graph of $y = (x - 1)^2 + 4$ is the same as the graph $y = x^2$ but is shifted 1 unit to the right and 4 units down.

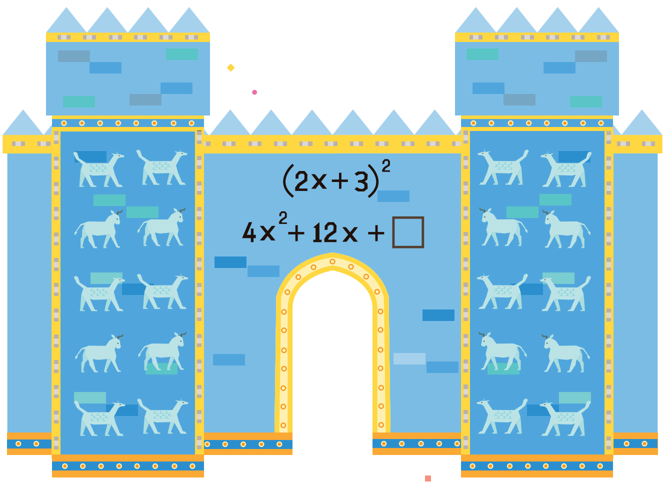
D. The graph of $y = (x - 1)^2 + 4$ is the same as the graph $y = x^2$ but is shifted 1 unit to the left and 4 units down.

5. Lin expanded the expression $(6x + 4)^2$ and determined the product is $36x^2 + 48x + 16$. Did she determine the product correctly? Explain or show your thinking.

Name: ______ Date: ______ Period: ______

Unit 6 | Lesson 15

Solving Non-Monic Quadratic Equations by Completing the Square



Let's solve other quadratic equations by completing the square.

Warm-up Find and Fix

Elena states, "The expression $(x + 3)^2$ can be rewritten as $x^2 + 6x + 9$. Therefore, the expression $(2x + 3)^2$ can be rewritten as $4x^2 + 6x + 9$."

Find and correct the error in Elena's statement. Explain or show your thinking.

Activity 1 Square in a Different Way

1. Each expression is written in $(mx + p)^2$ form. Use the given strategy to rewrite each expression in standard form, $ax^2 + bx + c$.

a $(4x + 3)^2$

Strategy: Area diagram

Standard form:

Describe the relationship between the values of m and p in factored form and the values of a and c in standard form.

b $(5x - 2)^2$

Strategy: Distributive Property

Standard form:

Describe the relationship between the values of m and p in factored form and the linear coefficient, b, in standard form.

c Rewrite $(3x + n)^2$ in standard form using any strategy.

d Rewrite $(mx + p)^2$ in standard form using any strategy.

2. True or false? The expression is a square expression.
   - If true, write the equivalent expression in factored form, $(mx + p)^2$.
   - If false, determine one term to change to make it a square expression. Explain your thinking.

a $4x^2 + 12x + 9$

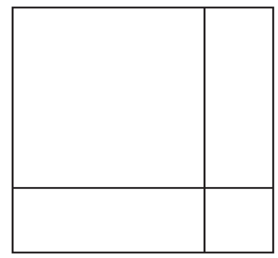
b $4x^2 + 8x + 25$

Name: ______ Date: ______ Period: ______

Activity 2 The Value of c

1. Consider the quadratic expression: $100x^2 + 80x + c$.

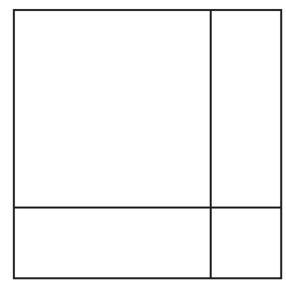
a Label the area diagram to determine the value of c that makes the expression a square expression. Then write it in standard form, $ax^2 + bx + c$.



b Rewrite your expression in part a in the form $(mx + p)^2$.

2. Consider the quadratic expression: $36x^2 - 60x + c$.

a Label the area diagram to determine the value of c to make the expression a square expression. Then write it in standard form, $ax^2 + bx + c$.



b Rewrite your expression in part a in the form $(mx + p)^2$.

3. Consider the quadratic expression: $25x^2 + 40x + c$. Determine the value of c that would make the expression a square expression. Then write the expression in the form $ax^2 + bx + c$ and in the form $(mx + p)^2$.

4. Solve each equation by completing the square.

a $25x^2 + 40x = -12$

b $36x^2 - 60x + 10 = -6$

Activity 3 Squaring a

Study each strategy for solving the quadratic equation $3x^2 + 8x + 5 = 0$.

Write in factored form.	Multiply, then substitute.	Complete the square.
$3x^2 + 8x + 5 = 0$ $(3x + 5)(x + 1) = 0$ $x = -\frac{5}{3}$ or $x = -1$	$3x^2 + 8x + 5 = 0$ $9x^2 + 24x + 15 = 0$ $(3x)^2 + 8(3x) + 15 = 0$ $N^2 + 8N + 15 = 0$ $(N + 5)(N + 3) = 0$ $N = -5$ or $N = -3$ $3x = -5$ or $3x = -3$ $x = -\frac{5}{3}$ or $x = -1$	$3x^2 + 8x + 5 = 0$ $9x^2 + 24x + 15 = 0$ $9x^2 + 24x + 16 = 1$ $(3x + 4)^2 = 1$ $3x + 4 = -1$ or $3x + 4 = 1$ $x = -\frac{5}{3}$ or $x = -1$

Solve each equation. Use each strategy at least once. Show your thinking.

1. $2x^2 + 6x - 20 = 0$

2. $8x^2 - 20x = -12$

Name: ______ Date: ______ Period: ______

Activity 3 Squaring a (continued)

3. $5x^2 + 17x + 6 = 0$

4. $12x^2 + 20x = 77$

5. $8x^2 - 26x = -21$

6. $6x^2 + 19x + 10 = 0$

Historical Moment

Babylonian Multiplication

Ancient Babylonian mathematicians used the following formulas to multiply two whole numbers, a and b:

$$ab = \frac{(a+b)^2 - a^2 - b^2}{2} \qquad ab = \frac{(a+b)^2 - (a-b)^2}{4}$$

1. Show, algebraically and with numerical examples, that both of these formulas are correct.

STOP

Summary

In today's lesson . . .

You saw that non-monic quadratic expressions that are also square expressions can be written in the form $(mx + p)^2$. You can also write square expressions in standard form by expanding them:

$(mx)^2 + 2(mx)(p) + p^2$ or $m^2x^2 + 2mpx + p^2$

If a quadratic square expression is already in standard form, $ax^2 + bx + c$, then:

- The value of a is m^2.
- The value of b is $2mp$.
- The value of c is p^2.

You can use this pattern when solving non-monic quadratic equations by completing the square.

Reflect:

Practice

Name: .... Date: .... Period: ....

1. Select *all* expressions that are square expressions.

A. $(7-3x)^2$

B. $4x^2+6x+\frac{9}{4}$

C. $9x^2+24x+16$

D. $2x^2+20x+100$

E. $(5x+4)(5x-4)$

F. $(1-2x)(-2x+1)$

2. Determine the missing value that makes the expression a square expression. Then write the expression in factored form.

a. $9x^2+42x+$ .... Factored expression: ....

b. $49x^2-28x+$ .... Factored expression: ....

c. $25x^2+110x+$ .... Factored expression: ....

d. $64x^2-144x+$ .... Factored expression: ....

e. $4x^2+24x+$ .... Factored expression: ....

3. Determine the value of c that makes each expression a square expression. Then write the expression in standard form and in factored form.

a. $4x^2+4x+c$

b. $25x^2-30x+c$

Practice

Name: ________ Date: ________ Period: ________

4. Solve each equation by completing the square. Explain or show your thinking.

a $4x^2 + 4x = 3$

b $25x^2 - 30x + 8 = 0$

5. Rewrite each quadratic function in vertex form.

a $f(x) = x^2 + 12x + 36$

b $g(x) = x^2 + 10x + 21$

c $h(x) = 2x^2 - 20x + 32$

6. Every irrational number lies between two consecutive whole numbers. For each of the following irrational numbers, what are those two whole numbers? Explain or show your thinking.

a $\sqrt{10}$

b $\sqrt{28}$

c $2 - \sqrt{12}$

SUB-UNIT

4 Roots and Irrationals

Narrative Connections

Where does a number call its home?

Whenever you add two whole numbers, such as 3 and 4, you get another whole number (in this case, 7). This works for *any two* whole numbers you add. Try as you might, it is impossible to get anything else. You can imagine whole numbers as a town, and addition as a bus that runs through it. It's fine for a while — you can see a lot this way — but eventually you realize you have never seen what lies beyond. For all you know, the town is enclosed by a fence, a great wall, or an abyss. If you never try to leave, you will never know.

But then, along comes a new operation: subtraction. If you subtract 5 from 10, you stay within the town's limits. But what happens if you try to subtract 10 from 5? Suddenly, you have broken through into someplace different! Here, there are negatives and opposites. Along with positive integers, negative integers and zero come together to make up a bustling metropolis. But over time — with just adding, subtracting, and multiplying — you eventually realize you have never seen what is outside this metropolis.

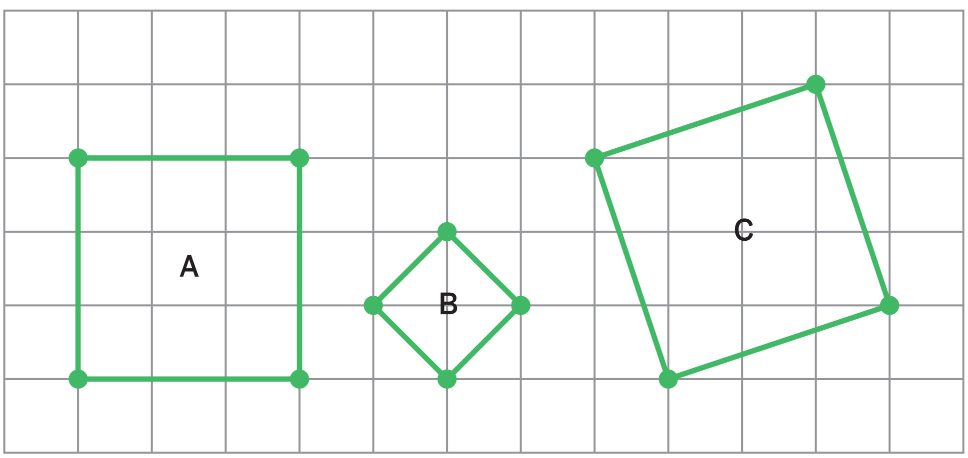
Ah, but what about division? Try splitting 5 into 3 equal parts, and you will see this metropolis of integers is just part of a larger country of rational numbers. But what is beyond this country? Take the square root of many rational numbers, such as 2, and you are out in a wider world that includes irrational numbers. Will this always be the case? And can you ever put these new numbers together to find your way back to where you started, with whole numbers, integers, and rationals?

Quadratic Equations With Irrational Solutions

Let's examine exact solutions to quadratic equations.

Warm-up Roots of Squares

Consider each square on the grid.



Complete the table with the missing information.

Figure	Area (square units)	Side length (units)
Square A		
Square B		
Square C		

Log in to Amplify Math to complete this lesson online.

Name: ______ Date: ______ Period: ______

Activity 1 Square Root Solutions

Thousands of years before the calculator was invented, ancient Chinese, Babylonians, and Indians each developed methods for approximating square roots with great accuracy. Separated by thousands of miles, they had no easy way to learn from one another or "look up" an answer! Study the two methods shown for approximating the square root of 41.

The **Babylonian method** begins with a guess.

Guess: The square root of 41 is 6.

$$\frac{6 + \frac{41}{6}}{2} \approx 6.42$$

$$\frac{6.42 + \frac{41}{6.42}}{2} \approx 6.4031$$

$$\frac{6.4031 + \frac{41}{6.4031}}{2} \approx 6.4031$$

Calculator check: $\sqrt{41} \approx 6.4031$

The **Indian method** involves finding a nearby square number. The greatest square number less than 41 is 36, whose square root is **6**, and 36 is **5** less than 41.

$$6 + \frac{5}{2(6)} - \frac{\left(\frac{5}{2(6)}\right)^2}{2\left(6 + \frac{5}{2(6)}\right)}$$

$$= \frac{77}{12} - \frac{\frac{25}{144}}{\frac{154}{12}}$$

$$= \frac{77}{12} - \frac{25}{144} \cdot \frac{12}{154} \approx 6.403$$

Calculator check: $\sqrt{41} \approx 6.403$

1. Use the Babylonian method to approximate $\sqrt{426}$. Show your thinking.

2. Use the Indian method to approximate $\sqrt{18}$. Show your thinking.

Critique and Correct: Your teacher will display an incorrect solution for either of these problems. With your partner, determine and correct the error. Explain how and why you corrected it.

Activity 1 Square Root Solutions (continued)

Ancient mathematicians were mostly curious about *positive* square roots, as they were useful in calculating the side lengths of squares. Nevertheless, there are two solutions to the equation $x^2 = 25$. Both $x = -5$ and $x = 5$ are solutions, because the equations $(-5)^2 = 25$ and $(5)^2 = 25$ are both true.

3. Solve each equation. Use ± (the plus-or-minus sign) when appropriate.

a $x^2 - 13 = -12$

b $(x - 6)^2 = 0$

c $x^2 = -8$

d $x^2 - 10 = 0$

e $x^2 + 1 = 18$

f $(x + 1)^2 = 18$

Historical Moment

The Square Root of 2

Ancient Babylonian mathematicians were able to determine the square root of 2 to six digits. This tablet, known as YBC 7289, depicts an accurate representation of the square root of 2 from around 1700 BCE.

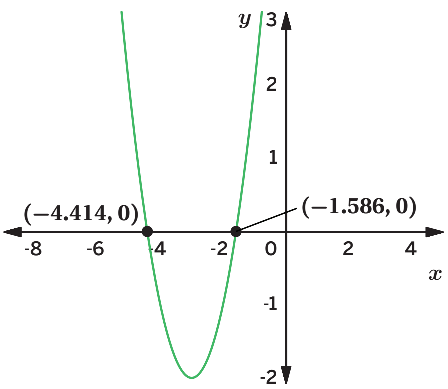
Use the Babylonian method to determine the first eight digits of $\sqrt{2}$. Begin with a guess of 1 and show your thinking. You may use a calculator, but *only* for adding, subtracting, multiplying, and dividing.

Courtesy of the Peabody Museum of Natural History, Division of Anthropology, Babylonian Collection, Yale University; http://peabody.yale.edu

Name: ______ Date: ______ Period: ______

Activity 2 Irrational Solutions

Consider the strategies for solving the quadratic equation $x^2 + 6x + 7 = 0$.

<table>
<tr><th>Graphing</th><th>Completing the square</th></tr>
<tr><td>

Approximate Solutions:

$x \approx -4.414$ or $x \approx -1.586$</td><td>$$\begin{aligned} x^2 + 6x + 7 &= 0 \\ x^2 + 6x + 9 &= 2 \\ (x+3)^2 &= 2 \\ x + 3 &= \pm\sqrt{2} \\ x &= -3 \pm \sqrt{2} \end{aligned}$$

Exact Solutions:

$x = -3 - \sqrt{2}$ or $x = -3 + \sqrt{2}$</td></tr>
</table>

For each equation, *approximate* the solutions by using graphing technology to complete each table. Round to the nearest thousandths. Then determine the exact solutions by completing the square.

<table>
<tr><th>Equation</th><th>Solve by graphing.</th><th>Solve by completing the square.</th></tr>
<tr><td>1. $x^2 + 4x + 1 = 0$</td><td><table><tr><th>x</th><th>y</th></tr><tr><td></td><td>0</td></tr><tr><td></td><td>0</td></tr></table>

Approximate solutions:</td><td>**Exact solutions:**</td></tr>
<tr><td>2. $x^2 - 10x + 18 = 0$</td><td><table><tr><th>x</th><th>y</th></tr><tr><td></td><td>0</td></tr><tr><td></td><td>0</td></tr></table>

Approximate solutions:</td><td>**Exact solutions:**</td></tr>
</table>

Activity 2 Irrational Solutions (continued)

Equation	Solve by graphing.	Solve by completing the square.
3. $x^2 + 5x + \frac{1}{4} = 0$	<table><tr><th>x</th><th>y</th></tr><tr><td></td><td>0</td></tr><tr><td></td><td>0</td></tr></table> **Approximate solutions:**	**Exact solutions:**
4. $9x^2 + 24x + 14 = 0$	<table><tr><th>x</th><th>y</th></tr><tr><td></td><td>0</td></tr><tr><td></td><td>0</td></tr></table> **Approximate solutions:**	**Exact solutions:**

Are you ready for more?

Write a quadratic equation of the form $ax^2 + bx + c$ whose solutions are $x = 5 - \sqrt{2}$ and $x = 5 + \sqrt{2}$. Show or explain your thinking.

STOP

Name: ________ Date: ________ Period: ________

Summary

In today's lesson . . .

You explored quadratic equations with irrational solutions. These irrational solutions can be expressed as exact or approximate solutions. Graphing tools or graphing technology will show the *approximate* solutions — rounded to some number of decimal places — while solving the quadratic equation algebraically will provide the *exact* solutions.

Reflect:

Practice

Name: .... Date: .... Period: ....

1. Calculate the exact solutions for each equation. Write the solutions using $\pm$ notation.

a $x^2 = 144$

b $x^2 = 5$

c $4x^2 = 28$

d $x^2 = \frac{25}{4}$

e $2x^2 = 22$

f $7x^2 = 16$

2. Match each expression using $\pm$ notation with the two values indicated by the $\pm$ notation.

a 4 ± 1 .... -17 or 5

b $10 \pm \sqrt{4}$ .... $4 + \sqrt{2}$ or $4 - \sqrt{2}$

c -6 ± 11 .... 8 or 12

d $4 \pm \sqrt{10}$ .... 3 or 5

e $\sqrt{16} \pm \sqrt{2}$ .... $4 + \sqrt{10}$ or $4 - \sqrt{10}$

Name: .... Date: .... Period: ....

3. *Technology required*. For each equation, *approximate* the solutions by using graphing technology to complete each table. Round to the nearest thousandths. Then determine the *exact* solutions by completing the square.

Equation	Solve by graphing.	Solve by completing the square.
a $x^2 + 10x + 8 = 0$	<table><tr><th>x</th><th>y</th></tr><tr><td></td><td>0</td></tr><tr><td></td><td>0</td></tr></table> **Approximate solutions:**	**Exact solutions:**
b $x^2 - 4x - 11 = 0$	<table><tr><th>x</th><th>y</th></tr><tr><td></td><td>0</td></tr><tr><td></td><td>0</td></tr></table> **Approximate solutions:**	**Exact solutions:**

4. Which factored expression is equivalent to $30x^2 + 31x + 5$?

A. $(6x + 5)(5x + 1)$

B. $(5x + 5)(6x + 1)$

C. $(10x + 5)(3x + 1)$

D. $(30x + 5)(x + 1)$

5. Determine whether each sum or product is a rational or irrational number. Explain your thinking.

a $\sqrt{3} \cdot \sqrt{3}$

b $3 \cdot \sqrt{3}$

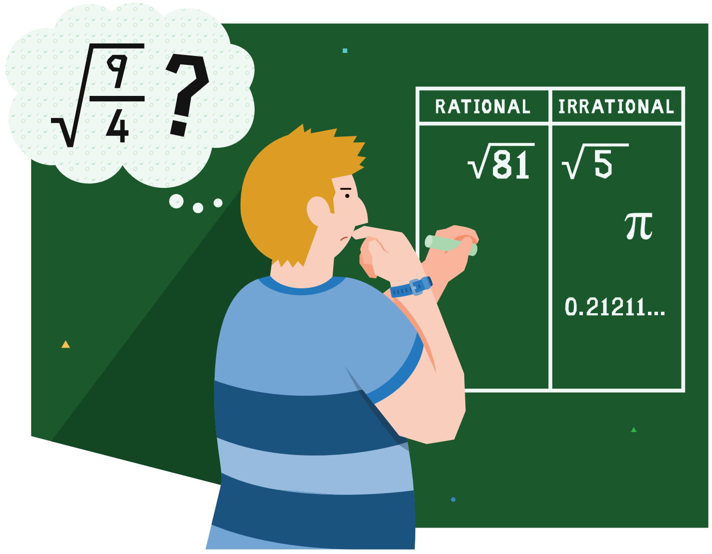
c $\sqrt{3} + \sqrt{3}$

d $3 + \sqrt{3}$

Unit 6 | Lesson 17

Rational and Irrational Numbers

Let's explore irrational numbers.



Warm-up Rational or Irrational?

Refer to the following list of numbers. Sort them in the table as rational or irrational numbers. If you are unsure about a specific number, sort it into the category "I am not sure."

97	-8.2	$\sqrt{8}$	$-\frac{3}{7}$	0	$\sqrt{\frac{9}{4}}$
$-\sqrt{18}$	$\sqrt{4}+\sqrt{9}$	$\sqrt{9}$	$\sqrt{2}$	$\frac{0}{2}$	$\sqrt{\frac{9}{2}}$

Rational number	Irrational number	I am not sure

Log in to Amplify Math to complete this lesson online.

Name: .... Date: .... Period: ....

Activity 1 Exploring Products and Sums

Consider the list of numbers.

2	3	$\frac{1}{3}$	0	$\sqrt{2}$	$\sqrt{3}$	$-\sqrt{3}$	$\frac{1}{\sqrt{3}}$

1. Choose two different numbers from the list. Determine the sum and product of those numbers. Numbers: ....

a. Sum:

b. Product:

2. What do you notice about the sum and product?

3. Repeat this process at least four more times, using a different set of numbers each time.

Numbers:	Numbers:	Numbers:	Numbers: ....
a. Sum:	a. Sum:	a. Sum:	a. Sum:
b. Product:	b. Product:	b. Product:	b. Product:

4. Using your results from Problem 3, determine whether the following statements are *always true*, *sometimes true*, or *never true*.

a. The sum of two rational numbers is rational.

b. The sum of a rational number and an irrational number is irrational.

c. The sum of two irrational numbers is irrational.

d. The product of two rational numbers is rational.

e. The product of a rational number and an irrational number is irrational.

f. The product of two irrational numbers is irrational.

Activity 2 Sums and Products of Rational Numbers

1. Is each sum a rational number? Explain your thinking.

a $4 + 0.175 = 4.175$

b $-0.75 + \frac{14}{8} = -\frac{6}{8} + \frac{14}{8} = \frac{8}{8} = 1$

c $\frac{1}{2} + \frac{4}{5} = \frac{5}{10} + \frac{8}{10} = \frac{13}{10}$

d a is an integer: $\frac{2}{3} + \frac{a}{15} = \frac{10}{15} + \frac{a}{15} = \frac{10 + a}{15}$

2. Suppose $\frac{a}{b}$ and $\frac{c}{d}$ are fractions. This means that a, b, c, and d are integers, and b and d cannot be 0.

a Determine the sum of $\frac{a}{b}$ and $\frac{c}{d}$. Show your thinking.

b In your sum, are the numerator and denominator integers? Explain your thinking.

c Explain why $\frac{a}{b} + \frac{c}{d}$ is a rational number.

Name: ________ Date: ________ Period: ________

Activity 2 Sums and Products of Rational Numbers (continued)

3. Determine the product of $\frac{a}{b} \cdot \frac{c}{d}$. Explain why the product of two rational numbers $\frac{a}{b} \cdot \frac{c}{d}$ must be rational.

Are you ready for more?

Consider numbers that are of the form $a + b\sqrt{5}$, where a and b are whole numbers. Let's call such numbers *quintegers*.

Refer to these examples of *quintegers*:

$\mathbf{3 + 4\sqrt{5}\ (a = 3, b = 4)}$ $\mathbf{7 - 2\sqrt{5}\ (a = 7, b = -2)}$

$\mathbf{-5 + \sqrt{5}\ (a = -5, b = 1)}$ $\mathbf{3\ (a = 3, b = 0)}$

1. When two quintegers are added, will the result always be another quinteger? Explain your thinking or provide a counterexample.

2. When you multiply two quintegers, will the product always be another quinteger? Explain your thinking or provide a counterexample.

Activity 3 Sums and Products of Rational and Irrational Numbers

1. Consider the sum of $\sqrt{2} + \frac{a}{b}$. Are there values of a and b such that the sum results in a rational number? Explain your thinking.

2. Consider the product of $\sqrt{2} \cdot \frac{a}{b}$. Are there values of a and b such that the product results in a rational number? Explain your thinking.

3. Consider the product of $\sqrt{a} \cdot \sqrt{b}$. Are there values of a and b such that the product results in a rational number? Explain your thinking.

STOP

Name: ______ Date: ______ Period: ______

Summary

In today's lesson . . .

You explored the sums and products of rational and irrational numbers.

When adding two numbers:

- The sum of two rational numbers is always rational.
- The sum of a rational number and an irrational number is always irrational.
- The sum of two irrational numbers is sometimes irrational. It is rational when a number and its opposite are added, for example, $\sqrt{a} + (-\sqrt{a}) = 0$.

When multiplying two numbers:

- The product of two rational numbers is always rational.
- The product of a rational number and an irrational number is sometimes irrational. It is rational when the rational number multiplied is 0, for example, $0 \cdot \sqrt{a} = 0$.
- The product of two irrational numbers is sometimes irrational. It is rational when a number is multiplied by itself, for example, $\sqrt{a} \cdot \sqrt{a} = a$.

Reflect:

Practice

Name: .... Date: .... Period: ....

1. Refer to these sums and products of rational and irrational numbers. Select *all* sums or products that are rational.

A. $5+\sqrt{5}$

B. $2.6+7.2$

C. $\frac{1}{5}\cdot\sqrt{4}$

D. $-\sqrt{3}+\sqrt{3}$

E. $\sqrt{16}\cdot\sqrt{2}$

F. $\sqrt{3}+\sqrt{13}$

2. Consider the statement: "An irrational number multiplied by an irrational number always results in an irrational product." Select all the expressions that show this statement is false.

A. $\sqrt{4}\cdot\sqrt{5}$

B. $\frac{1}{\sqrt{5}}\cdot\sqrt{5}$

C. $\sqrt{3}\cdot\sqrt{12}$

D. $-\sqrt{7}\cdot\sqrt{7}$

E. $\sqrt{25}\cdot\sqrt{4}$

F. $\sqrt{\frac{2}{13}}\cdot\sqrt{\frac{8}{13}}$

3. Provide a counterexample that shows that each statement is false.

a. An irrational number multiplied by an irrational number always results in an irrational product.

b. A rational number multiplied by an irrational number never results in a rational product.

c. Adding an irrational number to an irrational number always results in an irrational sum.

Name: ______________ Date: ________ Period: ________

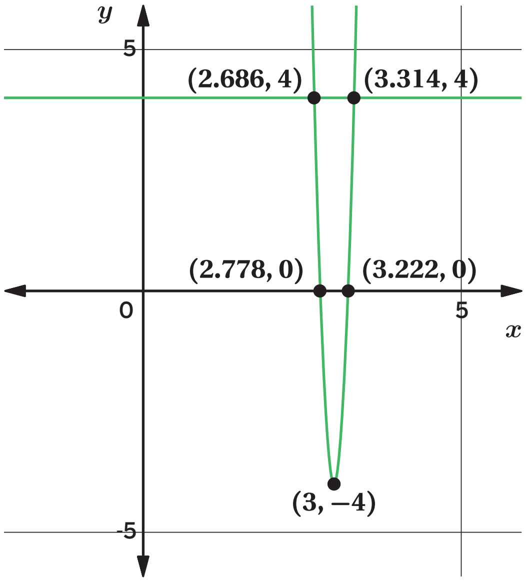
Practice

4. Refer to the quadratic equation $y = (x-3)^2 + 5$.

a Where is the vertex of the graph of this equation?

b Does the parabola open upward or downward? Explain your thinking.

5. Refer to the graph of the equations $y = 81(x-3)^2 - 4$ and $y = 4$.



a Based on the graph, what are the solutions to the equation $81(x-3)^2 = 4$?

b Based on the graph, what are the solutions to the equation $81(x-3)^2 - 4 = 4$?

6. Using your calculator, determine the value of $\frac{2}{\sqrt{2}}$ and $\sqrt{2}$.

a What do you notice about their values?

b Using your calculator, determine the products of these irrational numbers.

$\sqrt{2}$ and $\sqrt{2}$ $\sqrt{12}$ and $\sqrt{12}$ $\sqrt{7.5}$ and $\sqrt{7.5}$ $\sqrt{121}$ and $\sqrt{121}$

c What do you notice about the values of these products?

Unit 6 | Lesson 18

Rational and Irrational Solutions

Let's explore irrational solutions.

Warm-up Which One Doesn't Belong?

Which of the following numbers or expressions does not belong with the others? Explain your thinking.

A. $2.828427\ldots$

B. $\sqrt{8}$

C. $\dfrac{8}{\sqrt{8}}$

D. $\dfrac{8\sqrt{8}}{8}$

Log in to Amplify Math to complete this lesson online.

Name: .... Date: .... Period: ....

Activity 1 Exploring Irrational Denominators

1. Using a calculator, determine the decimal approximations of the irrational numbers from the Warm-up.

   a. $\sqrt{8}$

   b. $\frac{8}{\sqrt{8}}$

   c. $\frac{8\sqrt{8}}{8}$

2. Study the decimal approximation of all three numbers.

   a. What do you notice?

   b. Make a prediction about why this is the result.

3. Because the decimal approximations are all equal, it is true that $\frac{8}{\sqrt{8}} = \frac{8\sqrt{8}}{8}$.

   a. What fractional number would you need to multiply $\frac{8}{\sqrt{8}}$ by to result in $\frac{8\sqrt{8}}{8}$?

   b. Why does the value of $\frac{8}{\sqrt{8}}$ not change when you multiply by this number?

   c. Is it more straightforward to approximate the decimal value of $\frac{8}{\sqrt{8}}$ or $\frac{8\sqrt{8}}{8}$? Explain your thinking.

Activity 1 Exploring Irrational Denominators (continued)

The number $\frac{8}{\sqrt{8}}$ has an *irrational denominator* and the number $\frac{8\sqrt{8}}{8}$ has a *rational denominator*. To approximate the value of a number with an irrational denominator, you can *rationalize the denominator*.

4. Consider the number $\frac{5}{\sqrt{5}}$ with an irrational denominator of $\sqrt{5}$.

a Rationalize the denominator by multiplying by $\frac{\sqrt{5}}{\sqrt{5}}$. Show your thinking.

b Between which two whole numbers is $\frac{5}{\sqrt{5}}$ located? Explain your thinking.

c What value would you multiply $\frac{3}{\sqrt{7}}$ by to rationalize the denominator?

5. The table has expressions with irrational denominators. Complete the table by rationalizing the denominator of each expression. Then determine the two whole numbers between which each irrational number is located.

Irrational denominator expression	Rational denominator expression	Whole number values
$\frac{3}{\sqrt{3}}$		
$\frac{6}{\sqrt{2}}$		
$\frac{10}{\sqrt{5}}$		
$\frac{1}{\sqrt{12}}$		

Name: .... Date: .... Period: ....

Activity 2 Rational or Irrational Solutions?

Refer to these equations.

Equations	Prediction: Rational or Irrational x-intercepts?	x-intercepts	Prediction correct?
$y = x^2 - 8$			
$y = (x - 5)^2 - 4$			
$y = (x - 7)^2 - 2$			
$y = \left(\frac{x}{4}\right)^2 - 9$			

1. Study the structure and values within each equation. Complete the second column of the table by predicting whether each equation will have rational or irrational x-intercepts. Explain your thinking in the space below.

2. Graph each equation using graphing technology. Complete the the remaining two columns of the table by identifying the x-intercepts and whether your prediction was correct.

3. Consider the following equations.

$$x^2 - 8 = 0 \qquad (x - 5)^2 = 4 \qquad (x - 7)^2 - 2 = 0 \qquad \left(\frac{x}{4}\right)^2 - 9 = 0$$

a Determine the exact solutions to each equation algebraically.

b What about the structure and values of an equation resulted in an irrational solution?

c What is true about equations that have irrational solutions and their corresponding graphs?

Activity 2 Rational or Irrational Solutions? (continued)

4. Determine the x-intercepts of the equation $y = (x + 2)^2 - 10$. Explain how you can determine whether the x-intercepts are rational or irrational in two different ways.

Are you ready for more?

1. Predict whether the x-intercepts of the equation $y = \left(\frac{x}{\sqrt{3}}\right)^2 - 3$ will be rational or irrational.

2. Determine the solutions of the equation $\left(\frac{x}{\sqrt{3}}\right)^2 - 3 = 0$ using any method.

3. Was your prediction correct? Explain why or why not.

Name: ________ Date: ________ Period: ________

Activity 3 Equations With Irrational Solutions

1. Consider the equation $x^2 + bx + 4 = 0$.

a Complete the table by using the b-value to determine the number of solutions and whether the solutions are rational or irrational with that specific value of b.

b	Number of solutions	Rational or Irrational solutions
-9		
-6.5		
-5		
-4		
-2		
-1		
0		
$\frac{1}{2}$		
3		
4.1		
5		
8		

b Study your table. What do you notice?

c What do you wonder?

2. Consider the equation $4x^2 + bx + 9 = 0$.

a Determine a value of b so that the equation has two rational solutions.

b Determine a value of b so that the equation has two irrational solutions.

c Determine a value of b so that the equation has one solution.

d Determine a value of b so that the equation has no solutions.

Activity 3 Equations With Irrational Solutions (continued)

3. Using the equation $4x^2 + bx + 9 = 0$, describe the values of b that result in two, one, or no solutions. Explain your thinking.

	Two solutions	One solution	No solutions
Values of b			

4. For each scenario, write a quadratic equation that has the specified number of solutions. Explain your thinking for each equation.

a Two rational solutions.

b One rational solution.

c Two irrational solutions.

d No real solutions.

STOP

Name: ________________ Date: ________ Period: ________

Summary

In today's lesson . . .

You explored how to ***rationalize the denominator*** to approximate values of expressions with irrational denominators. To rationalize the denominator, you multiply by $\frac{\sqrt{a}}{\sqrt{a}}$, where $\sqrt{a}$ is the radical expression in the denominator, so that when simplified, there will no longer be an irrational denominator.

You then determined whether solutions to a quadratic equation were rational or irrational. When doing this, you explored when a quadratic equation has one, two, or no solutions.

- A quadratic equation has two real solutions when the graph of the equation intersects the x-axis twice. These solutions can be either rational or irrational numbers.
- A quadratic equation has one real solution when the graph of the equation intersects the x-axis once. This solution can be either a rational or irrational number.
- A quadratic equation has no real solution when it does not intersect the x-axis.

You will learn more about solutions that are not real in future mathematics courses.

Reflect:

Practice

Name: ______________ Date: ______________ Period: ______________

1. Rationalize each denominator to determine between which two whole numbers the irrational number is located.

a $\frac{2}{\sqrt{2}}$

b $\frac{10}{\sqrt{5}}$

c $\frac{18}{\sqrt{6}}$

d $\frac{8}{3\sqrt{3}}$

2. Solve each equation. Then determine whether the solutions are rational or irrational.

a $(x + 1)^2 = 4$

b $(x - 5)^2 = 36$

c $(x + 3)^2 = 11$

d $(x - 4)^2 = 6$

3. Fill in the boxes to create an equation that matches each description. Use the digits 1–9, no more than one time each.

a A quadratic equation with no solutions of the form $x^2 + \square x + \square = 0$.

b A quadratic equation with two solutions of the form $x^2 - \square = 0$.

c A quadratic equation with one solution of the form $x^2 - \square x + \square = 0$.

Practice

Name: ______ Date: ______ Period: ______

4. Determine whether each number is rational or irrational.

10 $\quad -3 \quad \frac{4}{5} \quad 3.\overline{33}$

$\sqrt{10} \quad \sqrt{4} \quad \frac{\sqrt{25}}{4} \quad \sqrt{0.6}$

5. Solve the quadratic equation $8x^2 - 26x = -21$ by writing the expression in factored from or by completing the square. Explain why you chose your method.

My Notes:

SUB-UNIT

5 The Quadratic Formula

Narrative Connections

What was the House of Wisdom?

In the late 8th century, while Europe was in the throes of its Dark Ages, the Middle East was a center of thought and knowledge. Under the rule of the Abbasid Caliphate, in what is now Iraq, a great library was built called the House of Wisdom.

Scholars accumulated knowledge from across the known world — from Babylonian, Greek, Jewish, Chinese, and Indian cultures and civilizations — translating texts of philosophy, astronomy, science, mathematics, and literature into Arabic. The House was a place where the ideas of Greek scholars such as Aristotle, Plato, and Euclid could crash together with Eastern thinkers such as Brahmagupta and Aryabhata. Overseeing it all was Muḥammad ibn Mūsā al-Khwārizmī.

Born around 780 CE, Al-Khwārizmī was one of House of Wisdom's first directors. Building on the works of Babylonian, Greek, Jewish, and Indian scholars, he wrote the first major treatise on algebra — a word that comes from the Arabic *al-jabr*, meaning "balance."

Unlike past mathematicians, Al-Khwārizmī presented math *systematically* — first providing rules for how the math behaved, then demonstrating how it worked. But Al-Khwārizmī's most influential achievement may have been his discovery of the quadratic formula — a skeleton key capable of unlocking *any* quadratic equation.

Unit 6 | Lesson 19

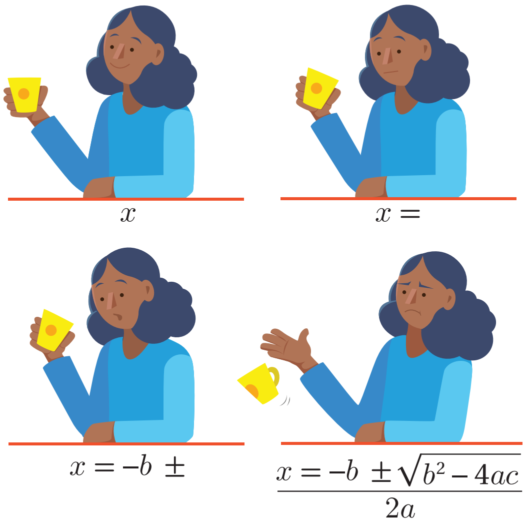
A Formula for Any Quadratic

Let's examine a new strategy for solving quadratic equations.

Warm-up Notice and Wonder

Consider the following meme.

What do you notice? What do you wonder?



1. I notice . . .

2. I wonder . . .

Log in to Amplify Math to complete this lesson online.

Name: ______ Date: ______ Period: ______

Activity 1 Deriving by Difference of Squares

Plan Ahead: How will you attend to the details of the derivation of the quadratic formula while also tracking the steps in the process?

Ancient Egyptians, Chinese, and Babylonians all came close to deriving the quadratic formula, which you will encounter later in this lesson. The Hindu mathematicians Brahmagupta and Baskhara made key advances around the year 700. Then, about 120 years later, Al-Khwārizmī derived the quadratic formula without using a single variable!

So how was it that Babylonian mathematicians got so close a thousand years earlier? They may have derived a similar quadratic formula by applying the difference of squares to quadratic equations. Let's follow in their footsteps.

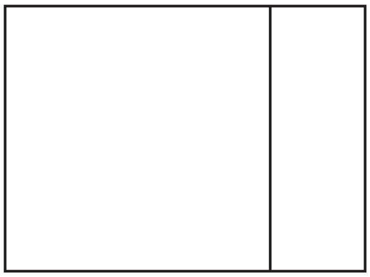
1. Consider the expression $(x + 3)(x - 3)$.

   a. Model the expression using an area diagram.

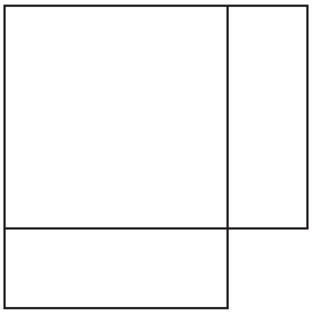
   b. Rewrite the expression $(x + 3)(x - 3)$ as a difference of squares.

2. Consider the expression $x(x + 8)$.

   a. Label the area diagram to model the expression.

   b. Label another area diagram, which has two equal linear terms, to model the same expression.

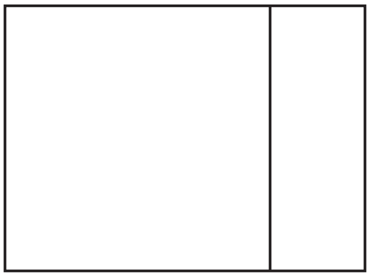
   c. Use your response from part b to rewrite the expression $x(x + 8)$ as a difference of squares.

   d. Use your response from part c to solve the equation $x^2 + 8x + 7 = 0$, using the difference of squares.

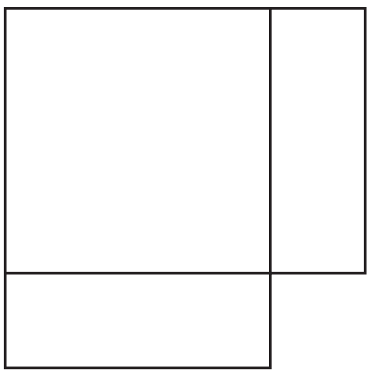
Activity 1 Deriving by Difference of Squares (continued)

3. Consider the expression $x^2 + bx$.

a Label the area diagram to model the expression.



b Label the new area diagram, which has two equal linear terms, to model the same expression.



c Use your response from part b to rewrite the expression $x^2 + bx$ as a difference of squares.

d Use your response from part c to solve the equation $x^2 + bx + c = 0$, using the difference of squares.

Name: .... Date: .... Period: ....

Activity 2 Deriving by Completing the Square

1. Determine whether factoring or completing the square would be a more efficient strategy to use when solving each equation. Explain your thinking. **Note:** You do not have to solve each equation.

a. $3x^2 + 24x + 21 = 0$

b. $x^2 + 6x + 7 = 0$

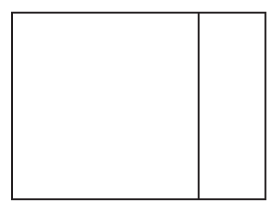
c. $4x^2 - 28x + 29 = 0$

You can solve any quadratic by completing the square. Sometimes this requires a lot of work. It would be great if there were another strategy with fewer steps. Let's start with the Babylonian geometric method of completing the square.

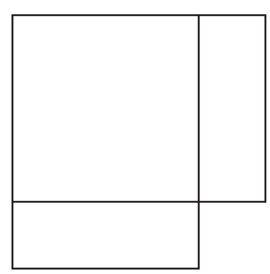
2. Begin with the standard form of a quadratic equation, $ax^2 + bx + c = 0$.

a. Divide each term by a. Then subtract the constant term from both sides of the equation. What equation is the result?

b. Label the area diagram to model the expression on the left side of your equation from part a.



c. Label this new area diagram, which has two equal linear terms, to model the same expression. What expression completes the square?



d. Add the expression you found in part c to both sides of the equation you wrote in part a. Simplify the right side of the equation.

Activity 2 Deriving by Completing the Square (continued)

e Rewrite the equation you wrote in part d by completing the square. Then solve the equation for x. If all goes well, you will have derived the ***quadratic formula***.

3. Refer to the equation in Problem 1c: $4x^2 - 28x + 29 = 0$. Its solutions can be determined using the quadratic formula: $x = \frac{28 \pm \sqrt{(-28)^2 - 4(4)(29)}}{2(4)}$. Explain how the values were substituted into the formula.

Are you ready for more?

The first several steps of an alternative method to derive the quadratic formula are shown. In this method, the first step is to multiply the expression $ax^2 + bx + c = 0$ by $4a$. Complete the remaining steps to show how the quadratic formula can be derived using this method.

$$\begin{aligned}
ax^2 + bx + c &= 0 \\
4a^2x^2 + 4abx + 4ac &= 0 \\
4a^2x^2 + 4abx &= -4ac \\
(2ax)^2 + 2b(2ax) &= -4ac \\
M^2 + 2bM &= -4ac \\
M^2 + 2bM + b^2 &= -4ac + b^2
\end{aligned}$$

STOP

Name: ________ Date: ________ Period: ________

Summary

In today's lesson . . .

You explored the derivation of the ***quadratic formula*** using two different methods, the difference of squares and completing the square.

The solutions of *any* quadratic equation written in standard form, $ax^2 + bx + c = 0$, can be determined using the quadratic formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Note that a, b, and c are values from the equation $ax^2 + bx + c = 0$, and a is not equal to 0. (If a were equal to 0, the equation would not be quadratic.)

Reflect:

Practice

Name: .... Date: .... Period: ....

1. Consider the quadratic equation $x^2 + 7x + 10 = 0$.

   a. Solve the equation by completing the square.

   b. The equation $x^2 + 7x + 10 = 0$ is written in standard form, $ax^2 + bx + c = 0$. What are the values of a, b, and c?

   c. Substitute the values of a, b, and c into the quadratic formula, $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, but do not evaluate any of the numerical expressions.

2. Consider the quadratic equation: $x^2 - 39 = 0$.

   a. Can you solve this equation using square roots? Explain or show your thinking.

   b. Can you solve this equation using the quadratic formula? Explain or show your thinking.

3. Clare derives the quadratic formula by completing the square for the equation $ax^2 + bx + c = 0$. She arrives at this step:

   $(2ax + b)^2 = b^2 - 4ac$

   Complete Clare's work by solving for x.

Name: ______ Date: ______ Period: ______

Practice

4. Match each expression in factored form with its equivalent expression in standard form.

Factored form	**Standard form**
a $(x+4)^2$	______ $9x^2 - 24x + 16$
b $(2x+5)^2$	______ $x^2 + 8x + 16$
c $(3x-4)^2$	______ $25x^2 + 30x + 9$
d $(5x+3)^2$	______ $4x^2 + 20x + 25$

5. Tyler solves the quadratic equation $x^2 + 8x + 11 = 4$. His work is shown. Describe the mistake he made. Then solve the equation correctly.

Tylers work:

$$x^2 + 8x + 11 = 4$$
$$x^2 + 8x + 16 = 4$$
$$(x+4)^2 = 2$$
$$x = -6 \text{ or } x = -2$$

6. Evaluate each expression.

a $\pm\sqrt{9} + 2$

b $\pm\frac{\sqrt{16}}{2}$

c $\pm\sqrt{(-2)^2 + 5}$

d $\pm\sqrt{42 - 3(2)}$

e -4 ± 23

Unit 6 | Lesson 20

The Quadratic Formula

Let's put the quadratic formula to work.

Warm-up Which One Doesn't Belong?

Which of these expressions does not belong with the others? Explain your thinking.

A. $1 \pm \sqrt{49}$

B. $\dfrac{\pm\sqrt{(-5)^2 - 4 \cdot 4 \cdot 1}}{3}$

C. $\dfrac{8 \pm 2}{5}$

D. $\dfrac{-18 \pm \sqrt{36}}{2 \cdot 3}$

Log in to Amplify Math to complete this lesson online.

Name: ______ Date: ______ Period: ______

Activity 1 Check It

The mathematician and Father of Algebra, Muḥammad ibn Mūsā al-Khwārizmī, derived a special formula to solve quadratic equations equal to 0. The quadratic formula can be used to calculate the solutions to any quadratic equation written in the form of $ax^2 + bx + c = 0$, where a, b, and c are values and a does not equal 0.

Consider the following worked example.

The solutions to the quadratic equation $x^2 - 8x + 15 = 0$ are $x = 5$ and $x = 3$.

Use the quadratic formula to show that the solutions are correct.

$a = 1$, $b = -8$, $c = 15$ →	• Determine the values of a, b, and c.
$x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ →	• Annotate the quadratic formula, color-coding the values a, b, and c.
$x = \dfrac{-(-8) \pm \sqrt{(-8)^2 - 4(1)(15)}}{2(1)}$ →	• Substitute the values of a, b, and c.
$x = \dfrac{8 \pm \sqrt{64 - 60}}{2}$ →	• Evaluate each part of the expression.
$x = \dfrac{8 \pm \sqrt{4}}{2}$ →	• Simplify.
$x = \dfrac{8 \pm 2}{2}$ →	• Take the square root.
$x = \dfrac{10}{2}$ or $x = \dfrac{6}{2}$ →	• Simplify.
$x = 5$ or $x = 3$ →	• Simplify.

1. Choose two of the following equations and identify a, b, and c in each of your chosen equations. Then substitute the values into the quadratic formula. You do not need to evaluate or simplify the formula.

a $x^2 + 4x - 5 = 0$

b $x^2 - 10x + 18 = 0$

c $9x^2 - 6x + 1 = 0$

d $6x^2 + 9x - 15 = 0$

Activity 1 Check It (continued)

2. The solutions for each quadratic equation are provided. Use the quadratic formula to show that the solutions are correct. Refer to the example at the start of this activity, if needed.

   a. Equation: $x^2 + 4x - 5 = 0$
   Solutions: $x = -5$ and $x = 1$

   b. Equation: $x^2 - 10x + 18 = 0$
   Solutions: $x = 5 - \frac{\sqrt{28}}{2}$ and $x = 5 + \frac{\sqrt{28}}{2}$

   c. Equation: $9x^2 - 6x + 1 = 0$
   Solution: $x = \frac{1}{3}$

   d. Equation: $6x^2 + 9x - 15 = 0$
   Solutions: $x = -\frac{5}{2}$ and $x = 1$

Featured Mathematician

Muḥammad ibn Mūsā al-Khwārizmī

Born in Persia in the 8th century, Al-Khwārizmī was known as the Father of Algebra. He studied and contributed to the fields of mathematics, astronomy, and geography. After presenting his geometric approach (i.e., completing the square), to solve quadratic equations, he went on to derive the quadratic formula. Along the way, he formalized algebra as a mathematical field, including how to solve equations.

Name: ______________________ Date: __________ Period: __________

Activity 2 Find and Fix

Choose two of the following equations and for each of your chosen equation, complete these tasks:

- **Solve the equation using the quadratic formula.**
- **Review the worked solution. Each worked solution contains at least one error.**
- **Find and describe any errors in the worked solution.**

	Worked solution, with error(s)	Describe the error(s)
1. $x^2 - 3x - 4 = 0$ **Correct solution(s):**	$x = \frac{-(-3) \pm \sqrt{(-3)^2 - 4(1)(-4)}}{2(1)}$ $x = \frac{3 \pm \sqrt{9 + 16}}{2}$ $x = \frac{3 \pm \sqrt{25}}{2}$ $x = \frac{3 + 5}{2}$ $x = 4$	
2. $2x^2 - 2x = 12$ **Correct solution(s):**	$2x^2 - 2x - 12 = 0$ $x = \frac{-2 \pm \sqrt{(-2)^2 - 4(2)(-12)}}{2(2)}$ $x = \frac{-2 \pm \sqrt{4 + 96}}{4}$ $x = \frac{-2 \pm \sqrt{100}}{4}$ $x = \frac{-2 \pm 10}{4}$ $x = \frac{-2 - 10}{4}$ or $x = \frac{-2 + 10}{4}$ $x = \frac{-12}{4}$ or $x = \frac{8}{4}$ $x = -3$ or $x = 2$	
3. $x^2 + 6x + 2 = 0$ **Correct solution(s):**	$x = \frac{-(6) \pm \sqrt{(6)^2 - 4(1)(2)}}{2(1)}$ $x = \frac{-6 \pm \sqrt{36 - 8}}{2}$ $x = \frac{-6 \pm \sqrt{28}}{2}$ $x = -6 \pm \frac{\sqrt{28}}{2}$ $x = -6 - \frac{\sqrt{28}}{2}$ or $x = -6 + \frac{\sqrt{28}}{2}$	

Activity 3 Choosing the Best Strategy

Quadratic equations have intrigued mathematicians for thousands of years. To solve these problems, different strategies were used.

- **In 1500 BCE, Egyptian mathematicians created tables to help calculate the areas of different squares and rectangles.**
- **By 400 BCE, Babylonian and Chinese mathematicians used geometric representations to complete the square.**
- **Between 700 CE and 1100 CE, Indian mathematicians determined the general solution(s) to the quadratic equation and later formalized that any positive number had two square roots.**
- **In 820 CE, Al-Khwārizmī derived the quadratic formula, without using a single symbol!**

Today, we still use different strategies to solve quadratic equations. Some strategies can always be used and some are simpler than others, but each one has advantages and disadvantages.

You will be given a card with one of four problems. For your assigned problem:

- Solve the equation using the given strategy.
- Explain how the assigned strategy compares to the other strategies.

When you return to your group:

- Describe your assigned equation and strategy, without focusing on the solutions.
- Discuss how your assigned strategy compares to the other strategies.

Are you ready for more?

1. Use the quadratic formula to write the solutions — as expressions — to the equation $ax^2 + c = 0$.

2. Solve the equation $3x^2 - 27 = 0$ in these two ways. Show your thinking.

   a. Without using any formulas.

   b. Using your expression from Problem 1.

3. Use the quadratic formula to write the solutions — as expressions — to the equation $ax^2 + bx = 0$.

4. Solve the equation $2x^2 + 5x = 0$ in these two ways. Show your thinking.

   a. Without using any formulas.

   b. Using your expression from Problem 3.

STOP

Name: .... Date: .... Period: ....

Summary

In today's lesson . . .

You saw that the quadratic formula can be used to find the solutions to any quadratic equation, $ax^2 + bx + c = 0$, including those that may be difficult or even impossible to solve using other strategies.

The quadratic formula: $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

At this point, the strategies you have learned to solve quadratic equations algebraically include:

- Factoring and using the Zero Product Principle.
- Completing the square.
- Using the quadratic formula.

For some quadratic equations, it may be more efficient to use one strategy than another. Knowing all of these strategies can help you choose the most efficient one to use, depending on the equation you are solving.

Reflect:

Practice

Name: ______ Date: ______ Period: ______

1. For each equation, identify the values of a, b, and c.

a. $3x^2 + 8x + 4 = 0$

b. $2x^2 - 5x + 2 = 0$

c. $-9x^2 + 13x - 1 = 0$

d. $x^2 + x - 11 = 0$

e. $-x^2 + 16x + 64 = 0$

2. The solutions for each quadratic equation are provided. Use the quadratic formula to verify that the solutions are correct.

a. Equation: $x^2 + 9x + 20 = 0$
Solutions: $x = -5$ or $x = -4$

b. Equation: $x^2 - 10x + 21 = 0$
Solutions: $x = 3$ or $x = 7$

c. Equation: $3x^2 - 5x + 1 = 0$
Solutions: $x = \dfrac{5 - \sqrt{13}}{6}$ or
$x = \dfrac{5 + \sqrt{13}}{6}$

3. For each equation, identify the values of a, b, and c.

a. $81 - x + 5x^2 = 0$

b. $\frac{4}{5}x^2 + 3x = \frac{1}{3}$

c. $121 = x^2$

d. $7x + 14x^2 = 42$

Name: ______________________ Date: ____________ Period: ____________

Practice

4. Select *all* equations that are equivalent to $81x^2 + 180x - 200 = 100$.

A. $(9x + 10)^2 = 0$

B. $(9x - 10)^2 = 10$

C. $(9x - 10)^2 = 20$

D. $(9x + 10)^2 = 400$

E. $81x^2 + 180x - 100 = 0$

F. $81x^2 + 180x + 100 = 200$

G. $81x^2 + 180x + 100 = 400$

5. *Technology required*. Two objects are launched upward. Each function gives the distance from the ground, in meters, as a function of time t, in seconds. Use graphing technology to graph each function.

Object A: $f(t) = 25 + 20t - 5t^2$ **Object B:** $g(t) = 30 + 10t - 5t^2$

a. Which object reaches the ground first? Explain your thinking.

b. What is the maximum height of each object?

6. Han solves the equation $x = -3 + \sqrt{3^2 - 4 \cdot 1 \cdot 2}$. His work is shown. Describe the mistake he made. Then solve the equation correctly.

Han's work:

$x = -3 + \sqrt{3^2 - 4 \cdot 1 \cdot 2}$

$x = -3 + 3 - 2 \cdot 1 \cdot 2$

$x = -4$

Unit 6 | Lesson 21

Error Analysis: Quadratic Formula

Let's analyze common mistakes made when using the quadratic formula.

Warm-up Bits and Pieces

Evaluate each expression for $a = 9$, $b = -5$, and $c = -2$.

1. $-b$

2. b^2

3. $b^2 - 4ac$

4. $-b \pm \sqrt{a}$

Log in to Amplify Math to complete this lesson online.

Name: ______________________ Date: __________ Period: __________

Activity 1 Find and Fix

For each equation, complete these tasks:

- Each worked solution uses the quadratic formula $\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.
- Review the worked solution. Each worked solution contains at least one error.
- Find and describe any errors in the worked solution.
- Provide the correct solution(s).

1. $2x^2 + 3 = 8x$

Worked solution

$$x = \frac{-(-8) \pm \sqrt{(-8)^2 - 4(2)(3)}}{2(2)}$$

$$x = \frac{8 \pm \sqrt{64 - 24}}{4}$$

$$x = \frac{8 \pm \sqrt{40}}{4}$$

$$x = 2 \pm \sqrt{10}$$

Error(s):

Corrected solution(s):

2. $x^2 + 3x = 10$

Worked solution

$$x = \frac{-3 \pm \sqrt{3^2 - 4(1)(10)}}{2(1)}$$

$$x = \frac{-3 \pm \sqrt{9 - 40}}{2}$$

$$x = \frac{-3 \pm \sqrt{-31}}{2}$$

No solutions

Error(s):

Corrected solution(s):

Activity 1 Find and Fix (continued)

3. $x^2 - 10x + 23 = 0$

Worked solution

$$x = \frac{-10 \pm \sqrt{(-10)^2 - 4(1)(23)}}{2}$$

$$x = \frac{-10 \pm \sqrt{-100 - 92}}{2}$$

$$x = \frac{-10 \pm \sqrt{-192}}{2}$$

Error(s):

Corrected solution(s):

4. $9x^2 - 2x - 1 = 0$

Worked solution

$$x = \frac{2 \pm \sqrt{(-2)^2 - 4(9)(-1)}}{2}$$

$$x = \frac{2 \pm \sqrt{4 + 36}}{2}$$

$$x = \frac{2 \pm \sqrt{40}}{2}$$

Error(s):

Corrected solution(s):

Name: ____________ Date: ____________ Period: ____________

Activity 2 Cannonballs and Ticket Prices

1. The function $g(t) = 50 + 312t - 16t^2$ represents the height, in feet, of a cannonball that was catapulted into the air as a function of time, in seconds.

   a. The cannonball reached a maximum height of $1{,}571$ ft. How many seconds after the launch did it reach its maximum height? Explain or show your thinking.

   b. Suppose a classmate is unconvinced by your solution. What is another way to show that your solution is correct?

2. The function $r(p) = 80p - p^2$ models the revenue a band expects to collect as a function of p, the price of one concert ticket. All amounts are in dollars. A band member says that a ticket price of either $\$15.50$ or $\$74.50$ would generate approximately $\$1{,}000$ in revenue. Do you agree? Explain or show your thinking.

Three Reads: Read Problem 2 three times to help you make sense of the scenario.

1. Understand the context.
2. Highlight given quantities.
3. Think about how you will respond.

Are you ready for more?

The function $f(t) = 2 + 30t - 5t^2 - 47$ has a graph that opens downward.

1. Determine the zeros of f without graphing. Explain or show your thinking.

2. Explain how the zeros found can be used to determine the vertex of the graph.

STOP

Summary

In today's lesson . . .

You saw how small errors using the quadratic formula can lead to an incorrect solution. Some common errors to avoid include:

- Using the incorrect values for a, b, or c in the formula.
- Forgetting to multiply a by 2 in the formula's denominator.
- Forgetting that squaring a negative number produces a positive number.
- Forgetting that the product of a negative number and a positive number is negative.

Reflect:

Practice

Name: .... Date: .... Period: ....

1. Andre and Bard are solving the equation $2x^2 - 7x = 15$ using the quadratic formula, but found different solutions. Study each student's work.

Andre's work:	**Bard's work:**
$x = \dfrac{-7 \pm \sqrt{7^2 - 4(2)(-15)}}{2(2)}$	$x = \dfrac{-(-7) \pm \sqrt{-7^2 - 4(2)(-15)}}{2(2)}$
$x = \dfrac{-7 \pm \sqrt{49 - (-120)}}{4}$	$x = \dfrac{7 \pm \sqrt{-49 - (-120)}}{4}$
$x = \dfrac{-7 \pm 13}{4}$	$x = \dfrac{7 \pm \sqrt{71}}{4}$
$x = -5$ or $x = \dfrac{3}{2}$	

a If this equation is written in standard form, $ax^2 + bx + c = 0$, what are the values of a, b, and c?

b Do you agree with either Andre or Bard? Explain your thinking.

2. The function $h(t) = -16t^2 + 80t + 64$ represents the height of a potato in feet, t seconds after it was launched from a mechanical device.

a Write an equation that represents when the potato will hit the ground.

b Determine the number of seconds it takes the potato to hit the ground without graphing. Show your thinking.

Practice

Name: ________ Date: ________ Period: ________

3. Priya determined that $x = 3$ and $x = -1$ are solutions to the equation $3x^2 - 6x - 9 = 0$. Is she correct? Explain your thinking.

4. Which of the following represents the solutions to the equation $2x^2 - 5x - 1 = 0$?

   A. $x = \frac{-5 \pm \sqrt{17}}{4}$

   B. $x = \frac{5 \pm \sqrt{17}}{4}$

   C. $x = \frac{-5 \pm \sqrt{33}}{4}$

   D. $x = \frac{5 \pm \sqrt{33}}{4}$

5. The data set and some statistics are listed:

   $11.5, 12.3, 13.5, 15.6, 16.7, 17.2, 18.4, 19, 19.5, 21.5$

   - mean: 16.52
   - median: 16.95
   - standard deviation: 3.11
   - IQR: 5.5

   a. How does adding 5 to each of the values in the data set impact the shape of the distribution?

   b. How does adding 5 to each of the values in the data set impact the measures of center?

   c. How does adding 5 to each of the values in the data set impact the measures of variability?

6. Evaluate the expression $2x^2 + 4x + c$ when $c = 8$ and $x = 3$.

Name: ________ Date: ________ Period: ________

Unit 6 | Lesson 22

Applying the Quadratic Formula

Let's use the quadratic formula to solve problems within a real-world context.

Warm-up Compare and Contrast

The work for solving two quadratic equations is shown.

Equation 1

$$(x+3)^2 - 9 = 0$$
$$(x+3)^2 = 9$$
$$x+3 = 3 \text{ or } x+3 = -3$$
$$x = 0 \quad \text{or} \quad x = -6$$

Equation 2

$$(x+3)^2 + 9 = 0$$
$$(x+3)^2 = -9$$
$$x+3 = \pm\sqrt{-9}$$

How are the two equations similar? How are they different?

Similarities:

Differences:

Log in to Amplify Math to complete this lesson online.

Activity 1 The Cannonball and the Pumpkin

Foofoo the Clown launched both a cannonball and a pumpkin from separate cannons to compare their paths.

The function $g(t) = 50 + 312t - 16t^2$ models the height, in feet, of a cannonball t seconds after it has been launched.

The function $h(t) = 2 + 23.7t - 4.9t^2$ models the height, in meters, of a pumpkin t seconds after it has been launched from the cannon.

1. After 8 seconds have passed, which object is still in the air? The cannonball or the pumpkin? Show your thinking.

2. Write equations for the pumpkin and the cannonball to determine when each object will hit the ground.

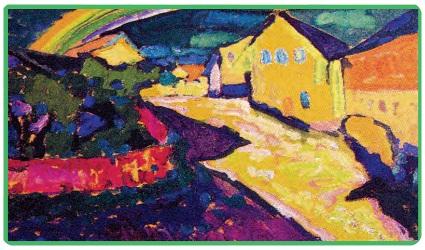
3. Use the quadratic formula to determine when each object hits the ground. Show your thinking.

4. Use the quadratic formula to determine when the cannonball will be 40 ft above the ground. Show your thinking.

Name: ______________ Date: ______________ Period: ______________

Activity 2 Picture Framing, Revisited

Earlier in this unit, you constructed a frame for a picture measuring 7 in. by 4 in. using a sheet of paper measuring 4 in. by 2.5 in. One equation you may have written to represent this scenario is $(7 + 2x)(4 + 2x) = 38$.



CEPTAP/Shutterstock.com

1. **a** Explain or show what the equation $(7 + 2x)(4 + 2x) = 38$ tells you about the scenario.

b What does x represent in this equation? Use a diagram, if helpful.

2. Rewrite the equation $(7 + 2x)(4 + 2x) = 38$ in standard form.

3. Solve your equation from Problem 3 using the quadratic formula. Show your thinking.

Activity 2 Picture Framing, Revisited (continued)

4. Suppose you have another picture that measures 10 in. by 5 in., and are now using a fancy paper that measures 8.5 in. by 4 in. to frame the picture. Again, the frame should be uniform in thickness all the way around the picture, and no fancy paper should be wasted. How thick should the frame be?

Are you ready for more?

Suppose the paper you use for a frame measures 6 in. by 8 in. You want to use all the paper to make a half-inch border around a rectangular picture.

1. What must be true about the length and width of any rectangular picture that can be framed this way?

2. Find two possible length and width pairs of a rectangular picture that could be framed with a half-inch border and no leftover material.

Name: ______ Date: ______ Period: ______

Activity 3 Beyond the Quadratic Formula

Just as the quadratic formula can be used to solve quadratic equations of the form $ax^2 + bx + c = 0$, the *cubic formula*, discovered by Gerolamo Cardano of Milan during the Italian Renaissance, can be used to solve cubic equations of the form $ax^3 + bx^2 + cx + d = 0$.

Here is the cubic formula:

$$x = \sqrt[3]{\left(\frac{-b^3}{27a^3} + \frac{bc}{6a^2} - \frac{d}{2a}\right) + \sqrt{\left(\frac{-b^3}{27a^3} + \frac{bc}{6a^2} - \frac{d}{2a}\right)^2 + \left(\frac{c}{3a} - \frac{b^2}{9a^2}\right)^3}}$$
$$+ \sqrt[3]{\left(\frac{-b^3}{27a^3} + \frac{bc}{6a^2} - \frac{d}{2a}\right) + \sqrt{\left(\frac{-b^3}{27a^3} + \frac{bc}{6a^2} - \frac{d}{2a}\right)^2 + \left(\frac{c}{3a} - \frac{b^2}{9a^2}\right)^3}} - \frac{b}{3a}$$

Your jaw may just have dropped — the cubic formula is *way* more complicated than the quadratic formula. But fear not! You will not be asked to use this formula. Instead, you will determine a cubic equation, given its solutions.

When solving cubic equations, Cardano would often manipulate them so they had the form $x^3 + ax + b = 0$.

1. You learned that quadratic equations can have 0, 1, or 2 solutions. What numbers of solutions do you think cubic equations can have?

2. The solutions to a specific cubic equation of the form $x^3 + ax + b = 0$ are $-2 - \sqrt{3}$, $-2 + \sqrt{3}$, and 4. Write the cubic equation in factored form.

3. Determine the original cubic expression $x^3 + ax + b$ by expanding your factored expression from Problem 2.

   a. First, multiply the two factors containing radical expressions and combine like terms.

   b. Multiply your expression by the third factor. Your resulting expression should be of the form $x^3 + ax + b$.

Reflect: In what ways can you show respect for the work Cardano did?

STOP

Summary

In today's lesson . . .

You saw that quadratic equations cannot always be neatly written in factored form or as a square expression. Completing the square will help you determine the solutions, but this strategy is often cumbersome. Graphing is also a helpful when solving quadratic equations, but often fails to give exact solutions.

With the quadratic formula, you can readily and precisely solve any quadratic equation precisely and efficiently.

Reflect:

Name: ______ Date: ______ Period: ______

Practice

1. Select *all* the equations that have two solutions.

A. $(x+3)^2 = 9$

B. $(x-5)^2 = -5$

C. $(x+2)^2 - 6 = 0$

D. $(x-9)^2 + 25 = 0$

E. $(x+10)^2 = 1$

F. $(x-8)^2 = 0$

G. $5 = (x+1)(x+1)$

H. $(x+1)^2 = 3$

2. A frog jumps in the air. Its height in inches is modeled by the function $h(t) = -16t^2 + 12t + 0.2$, where t is the time after it jumped, measured in seconds. Solve the equation $-16t^2 + 12t + 0.2 = 0$. What do the solutions tell you about the jumping frog?

3. A tennis ball is hit into the air and its height above the ground, in feet, is modeled by the function $f(t) = 4 + 12t - 16t^2$, where t is the number of seconds since the ball was hit.

a. What are the solutions to the equation $0 = 4 + 12t - 16t^2$?

b. What do the solutions tell you about the tennis ball?

4. Rewrite each quadratic expression in standard form.

a. $(x+1)(7x+2)$

b. $(8x+1)(x-5)$

c. $(2x+1)(2x-1)$

d. $(4+x)(3x-2)$

Practice

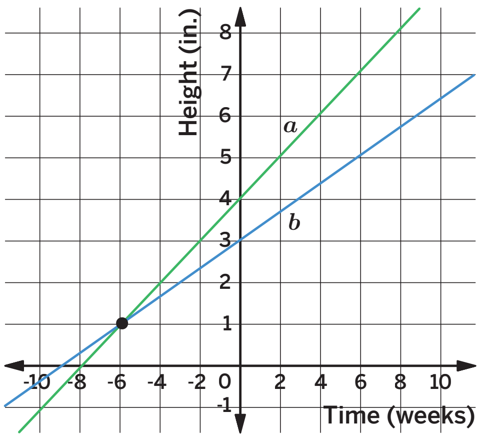
Name: ________ Date: ________ Period: ________

5. The number of downloads of a song w weeks since the song was released can be represented by the function $f(w) = 100000 \cdot \left(\frac{9}{10}\right)^w$.

a. What does the term $100{,}000$ tell you about the downloads? What about the term $\frac{9}{10}$?

b. Is $f(-1)$ meaningful in this situation? Explain your thinking.

6. Consider the following graph of the two plant heights.



a. Write an equation to model each line.

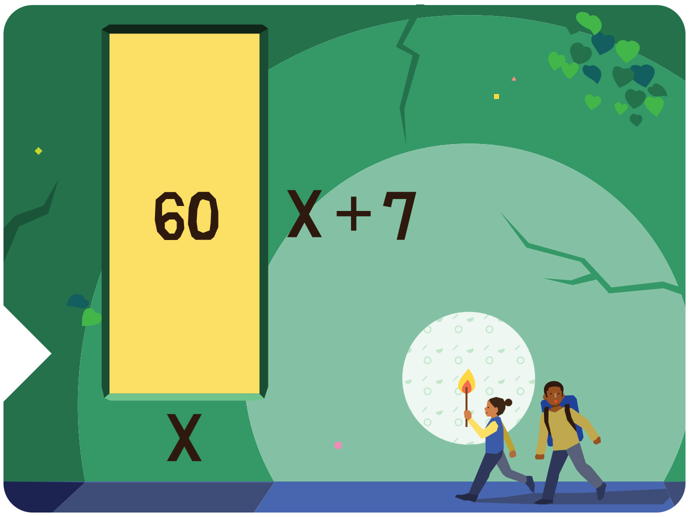
b. Determine the solution to the system of equations. Explain what the solution represents in this scenario.

Name: ________ Date: ________ Period: ________

Unit 6 | Lesson 23

Systems of Linear and Quadratic Equations

Let's find the intersections of the graphs of linear and quadratic functions.



Warm-up Math Talk

Discuss the strategies you would use to solve this ancient Babylonian problem. Then determine the solution.

The length of a rectangle exceeds its width by 7 units. Its area is 60 square units. Determine its length and width.

Strategy:

Solution:

Log in to Amplify Math to complete this lesson online.

Activity 1 A Babylonian Problem

Step into the shoes of an ancient Babylonian mathematician to solve this problem found on a clay tablet dated around 1700 BCE. Refer to the clay tablet.

"I have added the areas of my two squares; [result] 1,525. The side of the second square is $\frac{2}{3}$ the side of the first plus 5."

1. Write a system of equations to model the relationship between the first square with side length x and the second square with side length y.

2. Determine the values of x and y. Show your thinking.

Historical Moment

Nindas and Cubits

Consider the following Babylonian problem:

"I am using a reed to measure a rectangular plot of land with an area of 375 square nindas, but I do not know the reed's length. I broke off from it one cubit and walked 60 times along the plot's length. I restored to it what I have broken off, then walked 30 times along the plot's width. What was the original length of the reed?"

The *ninda* was a common unit for linear measurements of land, and was equivalent to 12 cubits. What was the original length of the reed in nindas?

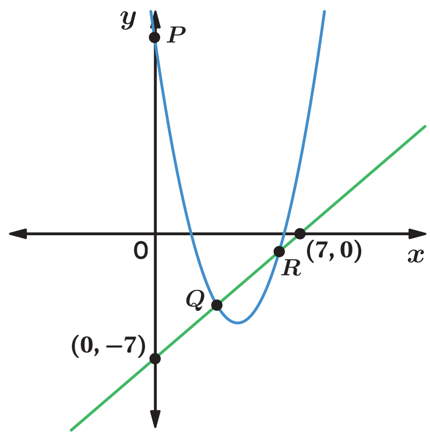
Show your thinking in the space provided.

Name: ______ Date: ______ Period: ______

Activity 2 Algebraic Connections

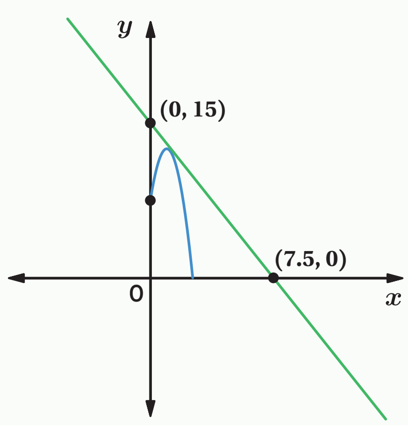
Consider the graph, which shows the quadratic function $f(x) = (x - 4)^2 - 5$, and a line passing through the points $(0, -7)$ and $(7, 0)$.

Determine the coordinates of points P, Q, and R without using graphing technology. Show your thinking.



Are you ready for more?

Consider the graph illustrating a zip line and the path of a diver jumping off a diving board, modeled by the equation $y = -5x^2 + 10x + 7.5$, where x represents the number of seconds after the diver jumps off the board and y represents the height, in meters, of the diver above the water.

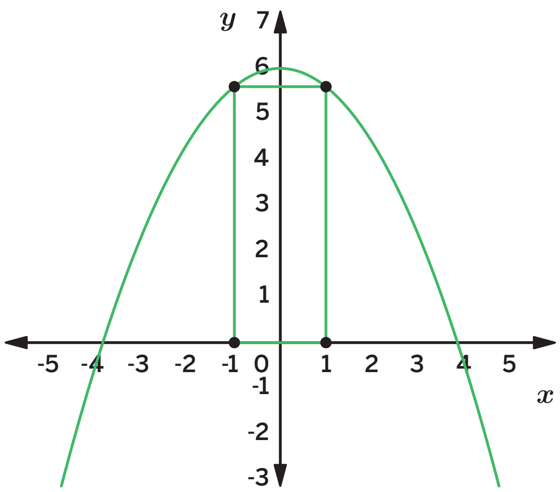


Will the diver hit the zip line? Explain your thinking.

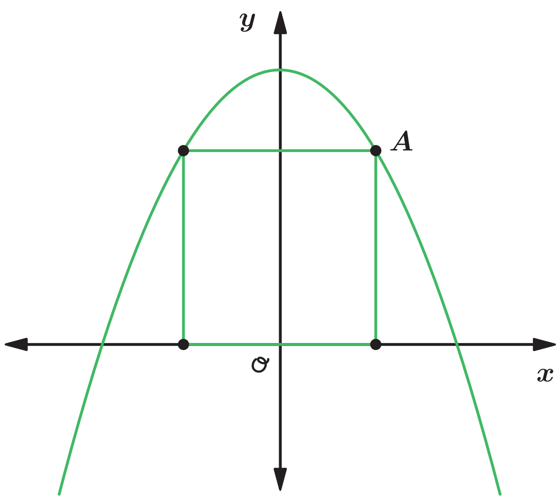
Activity 3 Geometric Connections

Consider the different rectangles inscribed between the x-axis and the quadratic function, $y = -0.4x^2 + 6$.

1. Two vertices of the rectangle shown are located at $(1, 0)$ and $(-1, 0)$. Determine the area of the rectangle. Show your thinking.



2. The figure shows a square inscribed between the x-axis and the quadratic function $y = -0.4x^2 + 6$.



a Kiran states he can use the equation $y = 2x$ to determine the coordinates of point A. Explain what this equation represents in terms of the square.

b Determine the area of the square. Show your thinking.

STOP

Name: ________________ Date: ________ Period: ________

Summary

In today's lesson . . .

You used linear and quadratic functions to solve and make sense of different real-world scenarios represented by mixed systems.

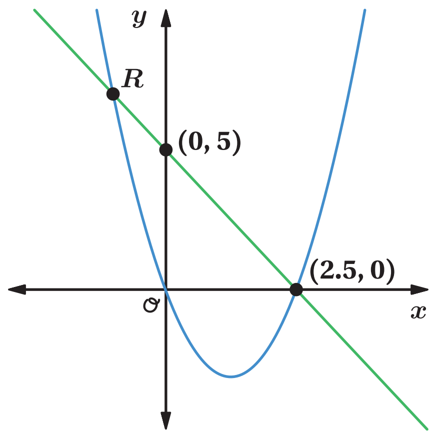
You modeled these scenarios with equations, interpreted graphs, and solved systems of linear and quadratic equations using different strategies.

Reflect:

Practice

Name: ______ Date: ______ Period: ______

1. The graph shows the quadratic function $f(x) = 2x^2 - 5x$, as well as the line passing through the points $(0, 5)$ and $(2.5, 0)$. Determine the coordinates of the point R without using graphing technology. Show or explain your thinking.



2. In this problem, you will write a system of linear and quadratic equations.

a. Write an equation representing the line that passes through the points $(0, -9)$ and $(9, 0)$.

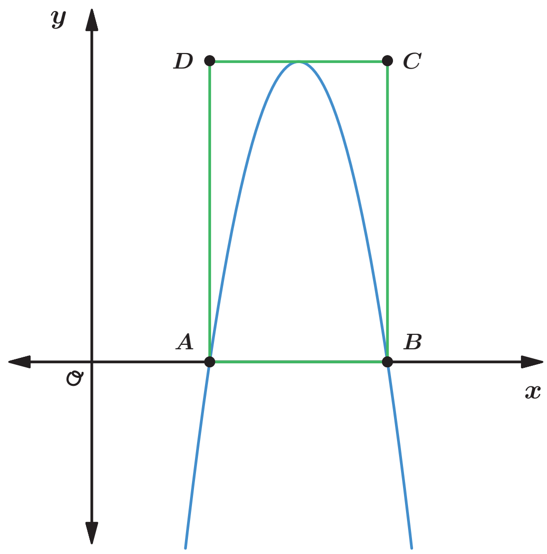
b. Write an equation representing a monic quadratic function that passes through the points $(-5, 0)$ and $(3, 0)$.

c. Without using graphing technology, determine the points of intersection of these two graphs. Show your thinking.

Name: ________ Date: ________ Period: ________

Practice

3. Consider the graph of the function $q(x) = -x^2 + 14x - 40$ and Rectangle $ABCD$. Points A and B are the x-intercepts of the graph and line segment CD touches the vertex of the parabola. Determine the area of Rectangle $ABCD$. Show or explain your thinking.



4. Sketch the graph of the quadratic function $h(x) = x^2 + 2x - 8$ without using graphing technology. Label the y-intercept, x-intercepts, and vertex.

5. Factor the quadratic expression $4m^2 - 81n^2$ using the difference of squares.

Unit 6 | Lesson 24 – Capstone

The Latest Way to Solve Quadratic Equations

Let's explore (yet another) way to solve quadratic equations.

Warm-up Difference of Squares

Mai factors a quartic expression using the difference of squares strategy.

Mai's work:

$$x^4 - 81$$

$$(x^2 + 9)(x^2 - 9)$$

Has Mai completely factored the expression? Explain or show your thinking.

Log in to Amplify Math to complete this lesson online.

Name: .... Date: .... Period: ....

Activity 1 Historical Origins

The solutions of a monic quadratic equation are p and q.

1. Write a possible quadratic equation in each form:

   a. Factored form

   b. Standard form

Use your equation from Problem 1b to complete Problems 2 and 3.

2. How does the coefficient of the linear term relate to the solutions p and q?

3. How does the constant term relate to the solutions p and q?

The French mathematician François Viète discovered that, for any equation of the form $x^2 + bx + c = 0$, the solutions p and q are determined by two numbers whose sum is $-b$ and whose product is c. Use Viète's discovery to complete each problem.

4. Consider the equation $x^2 - 8x + 12 = 0$.

   a. Without solving for x, what is the product of the solutions?

   b. Without solving for x, what is the sum of the solutions?

   c. What is the *average* of the two solutions? Explain your thinking.

   d. Write an expression to represent the average of the solutions of any quadratic equation of the form $x^2 + bx + c = 0$.

German Vizulis/Shutterstock.com

Thousands of years before Viète, Babylonian mathematicians realized that the linear term of a monic quadratic equation could be used to determine the average of its two solutions.

5. You can represent the two solutions as their average plus or minus u, where u represents the difference between each solution and their average.

   a. Use the average you calculated from Problem 4c to rewrite the solutions to the equation $x^2 - 8x + 12$ in terms of u.

      (........ $+ u$) and (........ $- u$)

   b. Verify the expressions in part a produce the average you determined in Problem 4c.

Activity 2 Making Math History in 2019

Po-Shen Loh, a math professor at Carnegie Mellon University, recently discovered an approach for solving quadratic equations based on Viète's and Babylonian mathematicians' observations. Let's follow in his footsteps and derive his strategy for ourselves.

Consider the equation $x^2 - 10x + 23 = 0$.

1. What is the product of its solutions?

2. What is the average of its solutions?

3. Use your average from Problem 2 to complete these problems.

- **a** Rewrite the solutions in terms of u, their difference from the average.
- **b** Write an equation that represents the product of these solutions.
- **c** Solve for u. Show your thinking.
- **d** Use the value you found for u to write a numerical expression for each solution.

4. You will now solve the equation $x^2 + x - 3 = 0$ on your own, using Po-Shen Loh's strategy.

- **a** What is the product of its solutions?
- **b** What is the average of its solutions?

Name: ____________ Date: ____________ Period: ____________

Activity 2 Making Math History in 2019 (continued)

c. Rewrite the solutions in terms of u, their difference from the average.

d. Determine the solutions of the equation $x^2 + x - 3 = 0$.

Note that this strategy specifically applies to monic quadratic equations. When $a \neq 1$, you can apply the properties of equality to determine an equivalent equation in which a *does* equal 1.

5. To solve the equation $9x^2 - 6x + 1 = 0$ using Po-Shen Loh's strategy:

a. Rewrite the equation $9x^2 - 6x + 1 = 0$ so that $a = 1$. Explain your process.

b. Determine the product and average of the solutions of the equation you wrote in part a.

c. Determine the solutions of $9x^2 - 6x + 1 = 0$.

Featured Mathematician

Po-Shen Loh

Po-Shen Loh is an associate professor of mathematics at Carnegie Mellon University, in Pittsburgh, Pennsylvania. His research lies at the intersection of combinatorics, probability, and computer science. Beyond his research and teaching, he has served as the head coach for the U.S. Math Olympiad Team, leading the U.S. to multiple first-place finishes in recent years. He has also founded several socially-minded startups, including Expii and Novid.

In 2019, Po-Shen announced his discovery of a new way to solve quadratic equations, demonstrating that creative thinking can lead to new developments, even when it involves math that is thousands of years old.

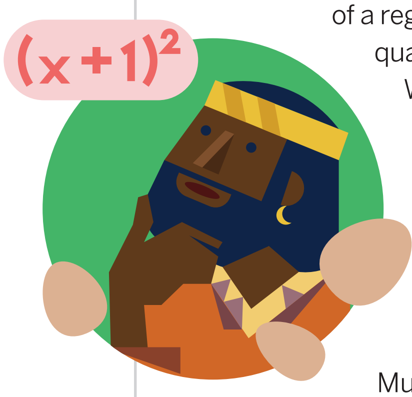
STOP

Unit Summary

You can solve most problems — both in math and out in the world — through different approaches. And, as you saw throughout this unit, quadratic equations are no different.

First, you journeyed back thousands of years to ancient Egypt, where the annual flooding of the Nile led to the development of a system for efficiently calculating the area of different plots of land.

Now some equations you can solve by factoring — but certainly not all of them. So we pick up our story almost a thousand years later in Mesopotamia and ancient China, who tackled quadratics through a method called "completing the square." It is a phrase that gets thrown around a lot in algebra courses. But it really is just that — finding the missing piece of a regular, bread-and-butter square. That way, the quadratic is changed into a simpler squared expression. Where there used to be two x's floating around in your equation, now there is just one!

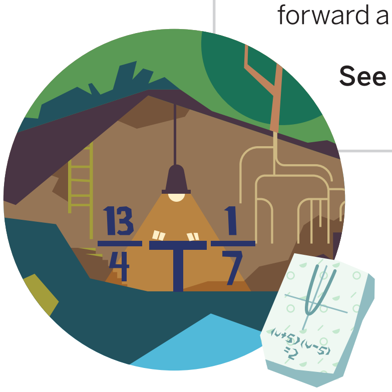


But the greatest triumph was still to come. With a firm foundation in Indian mathematics, Arab thinkers developed the quadratic formula — a skeleton key that may be a pain to memorize, but solves any quadratic equation for you in a pinch.

Much of this might look like ancient history, but it is not. It is the start of a long dialogue between different people, not least of all yourselves. Just today, you saw a modern cousin of the quadratic formula — Po-Shen Loh's method — which takes advantage of the symmetry between a quadratic equation's roots.

Math is more than just numbers. It is a conversation, an exchange of ideas. Some of these ideas took centuries to develop, surviving through conflict, war, and superstition. But in spite of it all, people learned from each other, incorporated each other's ideas, and moved forward a conversation that we continue to this day.

See you next year.



Practice

Name: .... Date: .... Period: ....

1. Bard uses Po-Shen Loh's strategy to solve the equation $1 - 8x + 16x^2 = 0$.

   a. What should be Bard's first step?

   b. What should Bard use for the average of the solutions?

   c. What value should Bard use for the product?

2. Use Po-Shen Loh's strategy to solve the equation $x^2 + 7x - 18 = 0$. Show your thinking.

Name: ______ Date: ______ Period: ______

Practice

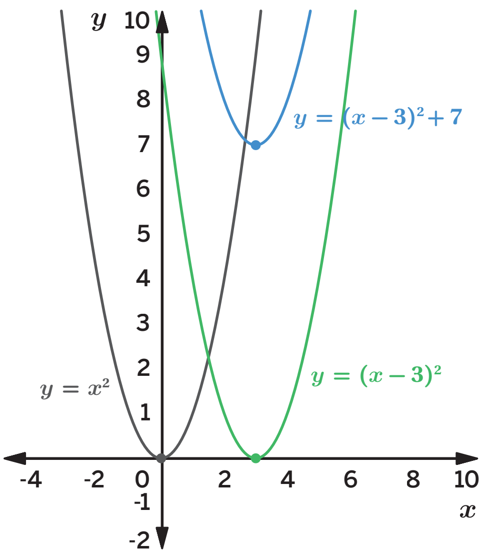
3. Match each quadratic equation with the number of solutions it has.

a $(x-1)(x-5)=5$ ______ no solutions

b $x^2-2x=-1$ ______ 1 solution

c $(x-5)^2=-25$ ______ 2 solutions

4. The graphs of the equations $y=x^2$, $y=(x-3)^2$, and $y=(x-3)^2+7$ are shown.



a How does the graph of $y=(x-3)^2$ compare to the graph of $y=x^2$?

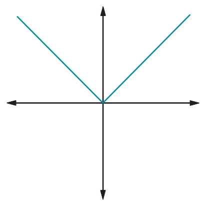
b How does the graph of $y=(x-3)^2+7$ compare to the graph of $y=(x-3)^2$?

Glossary/Glosario

A

English

absolute value function
A function whose output value is the distance of its input value from 0. In other words, the absolute value function is a piecewise function that takes negative input values and makes them positive.

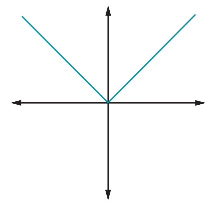


association When a change in one variable suggests another may change as well, the variables have an *association* and are said to be *associated* with one another.

average rate of change The ratio of the change in the outputs to the change in the inputs, for a given interval of a function.

Español

función de valor absoluto
Función cuya salida es la distancia entre su entrada y 0. En otras palabras, la función de valor absoluto es una función definida a trozos que toma entradas negativas y las hace positivas.



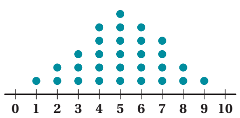
asociación Cuando un cambio en una variable sugiere que otra también podría cambiar, las variables tienen una *asociación* y están *asociadas* entre sí.

tasa de cambio promedio Razón entre el cambio de las salidas y el cambio de las entradas para un determinado intervalo de una función.

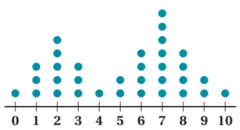
B

English

bell shaped A distribution that looks like a bell, with most of the data near the center and fewer points farther from the center, is called *bell shaped*.



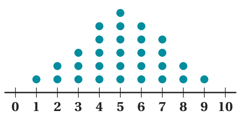
bimodal A distribution with two distinct peaks is called *bimodal*.



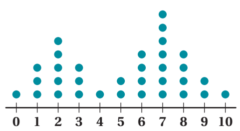
boundary line The line that represents the boundary between the region containing solutions and the region containing non-solutions for an inequality.

Español

acampanada Una distribución que asemeja a una campana, con la mayoría de los datos cerca del centro y una menor cantidad de puntos más lejos del centro, es llamada *acampanada*.



bimodal Una distribución con dos picos distintivos es llamada *bimodal*.



línea límite Línea que representa el límite entre la región que contiene soluciones a una desigualdad y la región que contiene no-soluciones.

Glossary/Glosario

English	Español
C	**C**
categorical variable A variable that can be partitioned into groups or categories.	**variable categórica** Variable que puede partirse en grupos o categorías.
causation When a change in one variable is shown, through careful experimentation, to cause a change in another variable.	**causalidad** Cuando se muestra que un cambio en una variable causa un cambio en otra variable, a través de cuidadosa experimentación.
common difference The difference between two consecutive terms in a linear pattern.	**diferencia común** Diferencia entre dos términos consecutivos de un patrón lineal.
common factor The factor by which each term is multiplied to generate an exponential pattern.	**factor común** Factor por el cual multiplicamos cada término para generar un patrón exponencial.
commutative property Changing the order in which numbers are either added or multiplied does not change the value of the sum or product.	**propiedad conmutativa** Cambiar el orden en que los números se suman o multiplican no cambia el valor de la suma o el producto.
completing the square Completing the square in a quadratic expression means transforming it into the form $a(x-h)^2+k$.	**completar el cuadrado** Completar el cuadrado en una expresión cuadrática significa transformarla en la forma $a(x-h)^2+k$.
compounding (interest) When interest itself earns further interest, it is said to be compounded, or applied to itself multiple times.	**(interés) compuesto** Cuando el interés genera más interés, se dice que es compuesto, o que se aplica a sí mismo múltiples veces.
constraint A limitation on the possible values of variables, often expressed by equations or inequalities. For example, distance above the ground d might be constrained to be non-negative: $d \geq 0$.	**limitación** Restricción de los posibles valores de las variables, usualmente expresada por ecuaciones o desigualdades. Por ejemplo, la distancia desde el suelo d puede ser limitada a ser no negativa: $d \geq 0$.
correlation coefficient A value that describes the strength and direction of a linear association between two variables. Strong positive associations have correlation coefficients close to 1, strong negative associations have correlation coefficients close to -1, and weak associations have correlation coefficients close to 0.	**coeficiente de correlación** Valor que describe la fuerza y dirección de una asociación lineal entre dos variables. Asociaciones positivas fuertes tienen coeficientes de correlación cercanos a 1, mientras que asociaciones negativas fuertes tienen coeficientes de correlación cercanos a -1, y asociaciones débiles tienen coeficientes de correlación cercanos a 0.
D	**D**
decay factor A common factor in an exponential pattern that is between 0 and 1.	**factor de decaimiento** Factor común en un patrón exponencial que se encuentra entre 0 y 1.
difference of squares Two squared terms that are separated by a subtraction sign.	**diferencia de cuadrados** Dos términos al cuadrado que están separados por un signo de resta.
discrete Separate and distinct values or points.	**discreto** Valores o puntos separados y distintivos.
discriminant For a quadratic equation of the form $ax^2+bx+c=0$, the discriminant is b^2-4ac.	**discriminante** Para una ecuación cuadrática de la forma $ax^2+bx+c=0$, el discriminante es b^2-4ac.
domain The set of all of possible input values for a given function.	**dominio** Conjunto de todos los posibles valores de entrada para una determinada función.

English

E

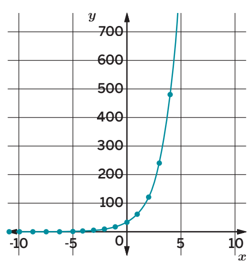
effective rate The actual interest amount earned over a year, taking into account the interest payment.

elimination The removal of a variable from a system of equations by adding or subtracting equations.

equivalent equations Equations that have the same solution or solutions.

equivalent systems Systems of equations that have the exact same solution or solutions.

exponential (growth) Describes a change characterized by the repeated multiplication of a common factor.



exponential function A one-to-one relationship in which a constant is raised to a variable power.

F

factored form (of a quadratic expression) A quadratic expression that is written as the product of a constant and two linear factors is said to be in factored form.

first difference The difference between two consecutive dependent terms for a function.

function notation A way of writing the output of a named function. For example, if the function f has an input x, then $f(x)$ denotes the corresponding output.

Español

E

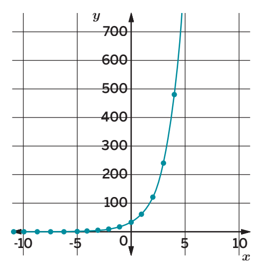
tasa efectiva Monto del interés real ganado en un año, después de tomar en cuenta el pago de intereses.

eliminación Anulación de una variable de un sistema de ecuaciones por medio de la suma o resta de ecuaciones.

ecuaciones equivalentes Ecuaciones que tienen la misma solución o soluciones.

sistemas equivalentes Sistemas de ecuaciones que tienen exactamente la misma solución o soluciones.

(crecimiento) exponencial Describe un cambio caracterizado por la multiplicación repetida de un factor común.



función exponencial Relación uno a uno en la cual una constante se eleva a una potencia variable.

F

forma factorizada (de una expresión cuadrática) Una expresión cuadrática escrita como el producto de una constante multiplicada por dos factores lineales se considera que está en forma factorizada.

primera diferencia Diferencia entre dos términos dependientes y consecutivos de una función.

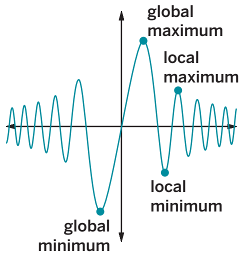
notación de función Forma de escribir la salida de una determinada función. Por ejemplo, si la función f tiene una entrada x, entonces $f(x)$ denota la salida correspondiente.

Glossary/Glosario

English | **Español**

G

global maximum The greatest value of a function over its entire domain.

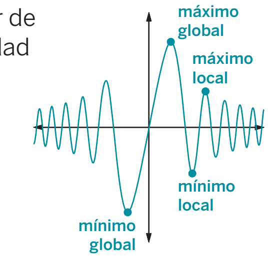


global minimum The least value of a function over its entire domain.

growth factor The common factor that is multiplied over equal intervals in an exponential pattern. In exponential functions of the form $f(x) = a \cdot (1 + r)^x$, the growth factor is $1 + r$.

growth rate The percent change of an exponential function. In exponential functions of the form $f(x) = a \cdot (1 + r)^x$, the growth rate is r.

máximo global El mayor valor de una función por sobre la totalidad de su dominio.



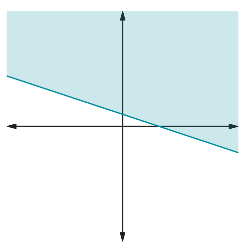
mínimo global El menor valor de una función por sobre la totalidad de su dominio.

factor de crecimiento Factor común que es multiplicado en intervalos iguales como parte de un patrón exponencial. En funciones exponenciales de la forma $f(x) = a \cdot (1 + r)^x$, el factor de crecimiento es $1 + r$.

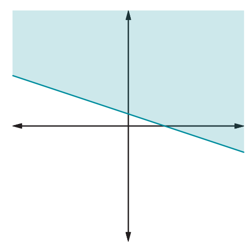
tasa de crecimiento El cambio porcentual de una función exponencial. En funciones exponenciales de la forma $f(x) = a \cdot (1 + r)^x$, la tasa de crecimiento es r.

H

half-plane The set of points in the coordinate plane on one side of a boundary line.



medio plano Conjunto de puntos en el plano de coordenadas que está a un solo lado de una línea límite.



I

index fund An investment fund constructed to track segments of a financial market.

infinity A boundless value, greater than that of any real number.

interest A percentage of the principal that is paid or owed over a specific amount of time.

interval notation A way to represent a set of numbers using parentheses and brackets. For example, the interval $(3, 5]$ represents all the values greater than 3 and less than or equal to 5.

inverse of a function The inverse of a function is created by reversing all of the function's input-output pairs. It can be determined by reversing the process that defined the original function.

fondo indexado Fondo de inversiones elaborado para seguir segmentos de un mercado financiero.

infinito Valor ilimitado, mayor que el valor de cualquier número real.

interés Un porcentaje del principal que se paga o debe durante un periodo de tiempo específico.

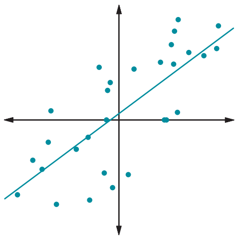
notación de intervalo Forma de representar un conjunto de números por medio de paréntesis y corchetes. Por ejemplo, el intervalo $(3, 5]$ representa todos los valores mayores que 3 y menores o iguales que 5.

inverso de una función El inverso de una función es creado al revertir todos los pares entrada-salida de la función. Se le puede determinar revirtiendo el proceso que definió a la función original.

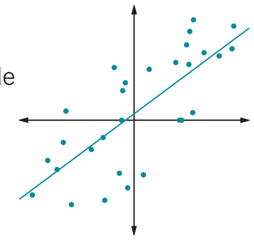
English	Español

L

line of best fit The linear model that has the smallest possible sum of the squares of the residuals.



línea de ajuste óptimo Modelo lineal que tiene la menor suma posible de los cuadrados de los residuos.



linear function A function with a constant rate of change.

función lineal Función con una tasa de cambio constante.

local maximum The value of a function that is greater than the nearby or surrounding values of the function.

máximo local Valor de una función que es mayor a los valores cercanos o circundantes de la función.

local minimum The value of a function that is less than the nearby or surrounding values of the function.

mínimo local Valor de una función que es menor a los valores cercanos o circundantes de la función.

M

monic quadratic An expression of the form $x^2 + bx + c$, where the coefficient of the x^2 term is 1.

ecuación cuadrática mónica Expresión de la forma $x^2 + bx + c$, en la cual el coeficiente del término x^2 es 1.

N

nominal rate The stated or published rate.

tasa nominal Tasa declarada o publicada.

non-monic quadratic An expression of the form $ax^2 + bx + c$, where a does not equal 1 or 0.

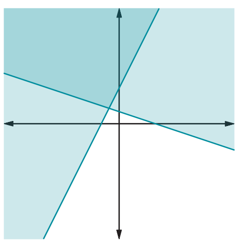
ecuación cuadrática no mónica Expresión de la forma $ax^2 + bx + c$, en la cual a no es igual a 1 o 0.

nonlinear relationship A relationship between two quantities in which there is no constant rate of change.

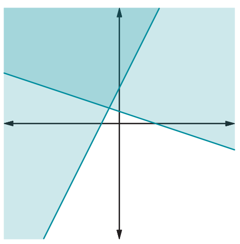
relación no lineal Una relación entre dos cantidades que no tiene una tasa de cambio constante.

O

overlap of graphs of inequalities The set of points that satisfy two or more inequalties.



superposición de gráficas de desigualdades Conjunto de puntos que satisfacen dos o más desigualdades.



Glossary/Glosario

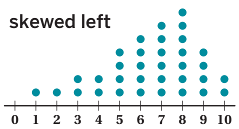
English	Español
P	**P**
piecewise function A function defined using different expressions for different intervals in its domain.	**función definida a trozos** Función definida por el uso de diferentes expresiones para diferentes intervalos de su dominio.
plus-or-minus symbol A symbol used to represent both the positive and negative of a number ($\pm$).	**símbolo de más menos** Usado para representar tanto el positivo como el negativo de un número ($\pm$).
principal Initial amount of a loan, investment, or deposit.	**principal** Monto inicial de un préstamo, inversión o depósito.
Q	**Q**
quadratic equation An equation in which the highest power of the variable is 2. Also called an equation of the second degree.	**ecuación cuadrática** Ecuación en la cual la potencia más alta de la variable es 2. También se llama ecuación de segundo grado.
Quadratic Formula The formula $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ that gives the solutions to the quadratic equation $ax^2 + bx + c = 0$, where $a \neq 0$.	**Fórmula cuadrática** Fórmula $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ que provee las soluciones de la ecuación cuadrática $ax^2 + bx + c = 0$, en la cual $a \neq 0$.
quadratic function A function in which the output is given by a quadratic expression.	**función cuadrática** Función en la cual la salida está dada por una expresión cuadrática.
R	**R**
range In algebra, a function's *range* is the set of all possible output values for the function. In statistics, the *range* of a data distribution is the difference between the maximum and minimum occurring values.	**rango** En algebra, el *rango* de una función es el conjunto de todos los posibles valores de salida de la función. En estadística, el *rango* de una distribución de datos es la diferencia entre los valores máximo y mínimo existentes.
relative frequency table A two-way table that shows the proportion of each value — expressed as fractions, decimals, or percentages — compared to the total in each row, column or in the entire table.	**tabla de frecuencia relativa** Tabla de doble entrada que muestra la proporción de cada valor (expresada como fracciones, decimales o porcentajes), en comparación con el total de cada fila, columna o con toda la tabla.
residual The difference between the actual y-coordinate and y-coordinate predicted by a model, given the x-coordinate.	**residuo** Diferencia entre la coordenada y real y la coordenada y pronosticada por un modelo, dada la coordenada x.
revenue The income generated from selling of a product or service.	**ingreso** Entrada de dinero generada por la venta de un producto o servicio.

English

S

second difference The difference between two consecutive first differences.

skewed A distribution with a long tail, where data extends far away from the center, is called *skewed*.



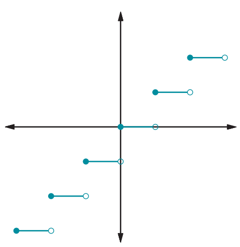
solution set The set of all values that satisfy an equation or inequality.

square expression An expression that represents the product of two identical expressions.

standard deviation A commonly used measure of variability. It is the square root of the average of the squares of the distances between data values and the mean.

standard form (of a quadratic expression) The standard form of a quadratic expression in x is $Ax^2 + Bx + C$, where A, B, and C are constants, and $A \neq 0$.

step function A piecewise function whose pieces are all constant values.



system of linear inequalities Two or more inequalities that represent the constraints in the same situation.

T

two-way table A table that organizes categorical data into cells. The categories do not overlap, so that each data value is recorded in exactly one cell.

U

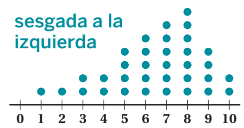
uniform A distribution in which data is evenly distributed throughout the range is called *uniform*.

Español

S

segunda diferencia Diferencia entre dos primeras diferencias consecutivas.

sesgada Una distribución de cola larga, en la cual los datos se extienden en dirección opuesta al centro, se conoce como *sesgada*.



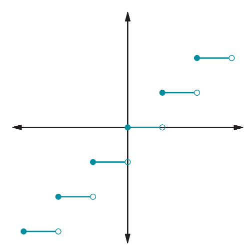
conjunto de soluciones Conjunto de todos los valores que satisfacen una ecuación o una desigualdad.

expresión cuadrada Expresión que representa el producto de dos expresiones idénticas.

desviación estándar Medida de variabilidad de uso común. Se trata de la raíz cuadrada del promedio de las distancias elevadas al cuadrado entre los valores de los datos y la media.

forma estándar (de una expresión cuadrática) La forma estándar de una expresión cuadrática en x es $Ax^2 + Bx + C$, en la cual A, B y C son constantes, y $A \neq 0$.

función escalonada Función definida a trozos, cuyos trozos son todos valores constantes.



sistema de desigualdades lineales Dos o más desigualdades que representan las limitaciones en la misma situación.

T

tabla de doble entrada Tabla que organiza datos categóricos en celdas. Las categorías no se superponen, de manera que el valor de cada dato es registrado exactamente en una sola celda.

U

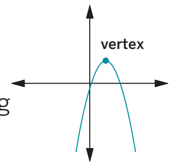
uniforme Distribución en la cual los datos son distribuidos de manera regular a través del rango.

Glossary/Glosario

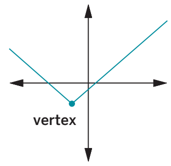
English

V

vertex (of a graph) The *vertex* of the graph of a quadratic function or of an absolute value function is the point where the graph changes from increasing to decreasing or vice versa. It is the highest or lowest point on the graph.



vertex form An equation of the form $y = a(x - h)^2 + k$ where (h, k) represents the coordinates of the vertex of a quadratic function.



Z

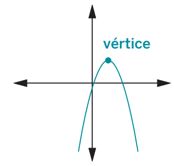
Zero Product Principle This principle states that $a \cdot b = 0$, if and only if $a = 0$ or $b = 0$.

zeros (of a function) The values at which the function is zero.

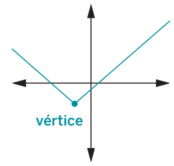
Español

V

vértice (de una gráfica) El *vértice* de la gráfica de una función cuadrática o de una función de valor absoluto es el punto en que la tendencia de la gráfica cambia de aumentar a disminuir o viceversa. Es el punto más alto o más bajo de la gráfica.



forma de vértice Ecuación de la forma $y = a(x - h)^2 + k$, en la cual (h, k) representa las coordenadas del vértice de una función cuadrática.



Z

Principio de producto cero Este principio establece que $a \cdot b = 0$ si y solo si $a = 0$ o $b = 0$.

ceros (de una función) Valores para los cuales la función es cero.

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