A Familiar Pattern

Let's look at different types of patterns.

Focus

Goals

1. Determine the missing terms in an arithmetic sequence.
2. Determine the common difference of an arithmetic sequence.
3. **Language Goal:** Explain what it means for a sequence to be arithmetic. **(Speaking)**

Coherence

Today

Students investigate what makes a sequence arithmetic. They look for and make use of structure to determine the common difference between successive terms and determine missing terms in an arithmetic sequence **(MP7)**. Students differentiate between arithmetic and non-arithmetic sequences and use mathematical arguments to support their reasoning **(MP3)**. This lesson also gives students the opportunity to use precise language to describe the relationship between consecutive terms as they formally define *arithmetic sequence* **(MP6)**. Students revisit the definition for *common difference* or *rate of change* and update their meaning in the context of sequences.

Previously

In Lesson 1, students studied patterns in a real-world situation and formally defined *sequences* and *terms*. Previously, students studied linear functions, and they continue to build on their understanding of linear patterns as they make connections to arithmetic sequences.

Coming Soon

In Lesson 3, students will investigate and define geometric sequences. They will continue to represent sequences in different ways using verbal descriptions, tables, and graphs.

Rigor

- Students build **procedural fluency** working with arithmetic sequences as they determine the common difference between terms and determine any missing terms in a sequence.

Standards

Addressing

HSF.LE.A.2
Construct linear and exponential **functions, including arithmetic** and geometric **sequences, given a graph, a description of a relationship, or two input-output pairs from a table.**

Building On	Building Toward
HSF.LE.A.1.A	HSF.LE.A.2
HSF.LE.A.2	HSF.BF.A.2

Pacing Guide

Suggested Total Lesson Time ~50 min

Warm-up	Activity 1	Activity 2	Activity 3	Summary	Exit Ticket
5 min	10 min	15 min	10 min	5 min	5 min
Independent	Pairs	Pairs	Pairs	Whole Class	Independent
MP7	MP6, MP7	MP3	MP6		
HSF.LE.A.2	HSF.LE.A.2	HSF.LE.A.2	HSF.LE.A.2	HSF.LE.A.2	HSF.LE.A.2

Amps powered by desmos | Activity and Presentation Slides

For a digitally interactive experience of this lesson, log in to Amplify Math at **learning.amplify.com**.

Practice Independent

Materials

- Exit Ticket
- Additional Practice
- Anchor Chart PDF, *Arithmetic Sequences*

Math Language Development

New words

- ***arithmetic sequence***
- ***common difference***

Review words

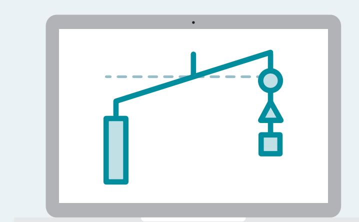
- *rate of change*
- *sequence*
- *slope*
- *term*

Amps | Featured Activity

Activity 1

Diagrams and Representations

Students use interactive tools to explore the progression of terms in a pattern.



Building Math Identity and Community

Connecting to Mathematical Practices

As students share their conclusions and strategies for distinguishing arithmetic sequences, they might not negotiate conflict constructively **(MP3)**. Prior to beginning the activity, discuss with the class the benefits of working with a partner. Have students identify ground rules for resolving conflict. Point out that sometimes, there is more than one possible pattern and that they both might be correct, but if not, there are always opportunities for both parties to learn from errors.

Modifications to Pacing

You may want to consider these additional modifications if you are short on time.

- In **Activity 1**, omit Pattern C in Problem 1.
- In **Activity 2**, reduce the number of sequences by omitting Sequences B and E.
- In **Activity 3**, omit Sequence A or B and its corresponding table in Problem 1 or 2. Answer Problems 4–6 using only one sequence.

Warm-up Notice and Wonder

MP7
HSF.LE.A.2

Students examine a dot pattern and notice how it changes to prepare them for a more formal approach to analyzing arithmetic sequences.

Unit 1 | Lesson 2

A Familiar Pattern

Let's look at different types of patterns.

Warm-up Notice and Wonder

Study the figures. What do you notice? What do you wonder?

Figure 1 Figure 2 Figure 3

1. I notice . . .
   Sample responses:
   - The figures are growing larger.
   - Each figure increases by four more dots.

2. I wonder . . .
   Sample responses:
   - Are there 13 dots in the next figure?
   - Does the pattern continue forever?

Collect and Display: Your teacher will highlight the math language you use to discuss the patterns and add it to a class display. Refer to the display during future discussions.

Log in to Amplify Math to complete this lesson online.

12 Unit 1 Sequences and Series

1 Launch

Conduct the ***Notice and Wonder*** routine and remind students there are no incorrect responses.

2 Monitor

Help students get started by having them compare each figure with the one before it **(MP7)**.

Look for points of confusion:

- **Not realizing the change in the number of dots is constant.** Ask students to write the number of dots below each figure and consider how they change.

Look for productive strategies:

- Drawing more figures to determine the pattern.
- Creating a table to extend the pattern numerically.

3 Connect

Have students share what they noticed and wondered and record their responses next to the pattern.

Display the dot pattern.

Highlight that the pattern increases by a constant amount each time.

Ask, "How does this growth pattern compare to others you have seen in Algebra 1?"

Math Language Development

MLR5: Co-craft Questions

During the Launch, have pairs of students craft 1–2 mathematical questions about the diagram as they wonder. Highlight questions that anticipate the number of dots in the next figures, such as, "Does Figure 4 have 13 dots? How many dots would Figure 100 have?"

English Learners: Give students time to craft their own questions and then use a think aloud strategy to model how to craft a mathematical question based on the pattern progression. This will help students foster metalinguistic awareness as they compare their questions to the modeled question.

Power-up

To power up students' ability to identify simple linear and exponential patterns, ask:

Assume each pattern continues. What is the next number in each pattern?

a. 5, 8, 11 14

b. 5, 15, 45 135

Use: Before the Warm-up

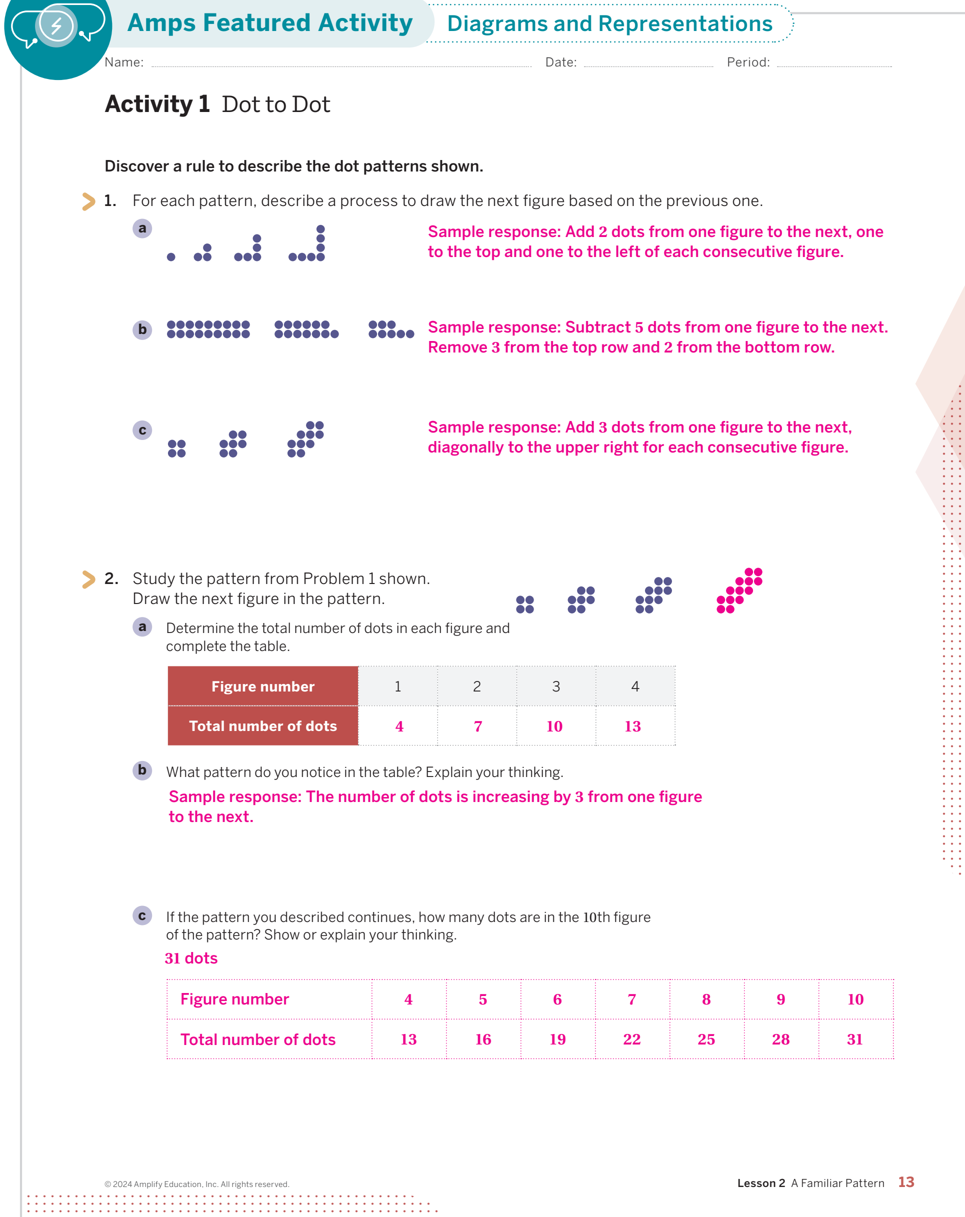
Informed by: Performance on Lesson 1, Practice Problem 6, and the Pre-Unit Readiness Assessment, Problems 1 and 5

Pairs | 10 min

MP6, MP7
HSF.LE.A.2

Activity 1 Dot to Dot

Students explore dot patterns and make connections to properties of arithmetic sequences.



Amps Featured Activity | Diagrams and Representations

Name: Date: Period:

Activity 1 Dot to Dot

Discover a rule to describe the dot patterns shown.

1. For each pattern, describe a process to draw the next figure based on the previous one.

   a. Sample response: Add 2 dots from one figure to the next, one to the top and one to the left of each consecutive figure.

   b. Sample response: Subtract 5 dots from one figure to the next. Remove 3 from the top row and 2 from the bottom row.

   c. Sample response: Add 3 dots from one figure to the next, diagonally to the upper right for each consecutive figure.

2. Study the pattern from Problem 1 shown. Draw the next figure in the pattern.

   a. Determine the total number of dots in each figure and complete the table.

Figure number	1	2	3	4
Total number of dots	4	7	10	13

   b. What pattern do you notice in the table? Explain your thinking.
   Sample response: The number of dots is increasing by 3 from one figure to the next.

   c. If the pattern you described continues, how many dots are in the 10th figure of the pattern? Show or explain your thinking.
   31 dots

Figure number	4	5	6	7	8	9	10
Total number of dots	13	16	19	22	25	28	31

1 Launch

Read the prompt as a class. Conduct the *Think-Pair-Share* routine.

2 Monitor

Help students get started by annotating the pattern in part a with successive differences.

Look for points of confusion:

- **Describing the patterns without using the common difference.** Ask, "What can you say about the change between each term in the sequence?" Students should respond verbally before writing their explanations.
- **Extending the visual pattern to determine the 10th term of the sequence in Problem 2.** Ask students to continue the sequence numerically.

Look for productive strategies:

- Drawing more figures to determine the patterns.
- Creating a table to extend the patterns in Problem 1 numerically.
- Using structure to analyze the pattern to determine the 10th term **(MP7)**.

3 Connect

Display the Anchor Chart PDF, *Arithmetic Sequences* and Problem 2.

Have pairs of students share their responses to Problem 2 and annotate the arithmetic sequence in the same format shown in the Anchor Chart PDF **(MP6)**.

Highlight that you need to calculate the *common difference* between successive terms to decide whether a sequence is arithmetic.

Define *arithmetic sequence* as a sequence in which each term is the previous term plus a constant, which is the ***common difference***.

Differentiated Support

Accessibility: *Optimize Access to Technology*

Have students use the Amps slides for this activity, in which they use interactive tools to explore the progression of terms in a pattern.

Extension: *Math Enrichment*

Have students complete Problem 2c for both Pattern A and Pattern B from Problem 1.
Sample response: Pattern A: 19; Pattern B: –27

Math Language Development

MLR2: Collect and Display

As students discuss the sequences, circulate and listen to students talk about the patterns they notice. Write down common or important phrases you hear students say about each type onto a large visual display, for example, "Add 10 each time." Add students' responses to the visual display. Throughout the remainder of the lesson and the entire unit, continue to update collected student language and remind students to use language from the display, as needed.

English Learners: As the terms *common difference* and *arithmetic sequence* are defined, add examples and annotations for each to the class display.

Pairs | 15 min

MP3
HSF.LE.A.2

Activity 2 Take Turns: Does It Stay the Same?

Students take turns distinguishing between different types of sequences to reinforce their understanding of the definition of an arithmetic sequence, and then use mathematical reasoning to support their conclusions **(MP3).**

Activity 2 Take Turns: Does It Stay the Same?

You and your partner will take turns completing Column 1 or Column 2 for each sequence.

Column 1 Determine the missing terms in the sequence, and then determine whether the sequence is arithmetic. If the sequence is arithmetic, state the common difference.	**Column 2** Actively listen and analyze your partner's response. Say and circle whether you agree or disagree, and then explain why. If you disagree, work together to reach an agreement.
Sequence A: 555, 455, 355, **255**, **155** **Arithmetic sequence?** Yes No **Common difference** (if applicable): **−100**	I agree / I disagree because . . . **you subtract 100 to get from one term to the next.**
Sequence B: 0.28, 0.30, 0.32, **0.34**, **0.36** **Arithmetic sequence?** Yes No **Common difference** (if applicable): **0.02**	I agree / I disagree because . . . **you add 0.02 to get from one term to the next.**
Sequence C: 32, 8, 2, $\frac{1}{8}$, $\frac{1}{2}$ **Arithmetic sequence?** Yes No **Common difference** (if applicable):	I agree / I disagree because . . . **you do not subtract but divide each term by 4.**
Sequence D: 1, 5, 25, **125**, **625** **Arithmetic sequence?** Yes No **Common difference** (if applicable):	I agree / I disagree because . . . **you do not add but multiply each term by 5.**
Sequence E: $\frac{1}{11}$, $\frac{2}{11}$, $\frac{3}{11}$, $\frac{4}{11}$, $\frac{5}{11}$ **Arithmetic sequence?** Yes No **Common difference** (if applicable): $\frac{1}{11}$	I agree / I disagree because . . . **you add $\frac{1}{11}$ to get from one term to the next.**

Reflect: How well did you and your partner resolve any conflicts? What did you learn from each other?

1 Launch

Read the prompt as a class. Conduct the ***Take Turns*** routine. Display the Anchor Chart PDF, *Arithmetic Sequences*, during the activity.

2 Monitor

Help students get started by having them annotate the terms of each sequence with their common difference, when possible.

Look for points of confusion:

- **Thinking a negative common difference should be positive.** Ask, "What does it mean for a common difference to be positive? Negative? Will a decreasing arithmetic sequence have a positive or negative common difference?"

Look for productive strategies:

- Explaining why each sequence is arithmetic or not by calculating the common difference.

3 Connect

Display all the sequences.

Have pairs of students share their conclusions and strategies for distinguishing arithmetic sequences. Select and sequence pairs to share how Sequences C and D are changing.

Highlight that arithmetic sequences increase or decrease by the same amount each term. If the common difference is positive, then you add the same number every term and the sequence increases. If the common difference is negative, then you subtract the same number every term and the sequence decreases.

Ask:

- "If you were only given the values of the 2nd and 5th terms of a sequence, how could you determine the common difference and calculate missing terms?"
- "For example, what are the missing values in this sequence? ____, 3, ____, ____, 18" −2, 8, 13; Sample response: Calculate the difference between 18 and 3 and divide by 3 because there are 3 terms from the 2nd to the 5th term. $18 - 3 = 15$ and $\frac{15}{3} = 5$. The common difference is 5.

Differentiated Support

Accessibility: *Vary Demands to Optimize Challenge*

If students need more processing time, have pairs of students focus on completing Sequences A–C and only work on Sequences D–G if they have time.

Extension: *Math Enrichment*

Have students complete the following problem:

What are the missing values in the following sequence?

___, ___, 5, ___, ___, 14,___

Sample response: −1, 2, 5, 8, 11, 14, 17

Math Language Development

MLR8: Discussion Supports

Use partner discussion to foster structured conversation as students complete Columns 1 and 2. As one student completes Column 1, stop students and ask them to turn and talk to their partner to determine whether they agree or disagree with each other's work completed in Column 1. Press students to come to an agreement before moving on to the next sequence.

English Learners: Encourage students to refer to and use the language and phrases from the class display as they turn and talk to each other.

Pairs | 10 min

Activity 3 What Does It Look Like?

MP6
HSF.LE.A.2

Students represent arithmetic sequences with verbal descriptions, tables, and graphs in order to make connections to linear functions.

Name: Date: Period:

Activity 3 What Does It Look Like?

You can also represent sequences using verbal descriptions, tables, or graphs. Here is a rule that can be used to build a sequence of numbers once a starting number is chosen: Each number is 3 less than the previous number.

1. Starting with the number 0, build a sequence of five numbers and complete the table.

Sequence A

Term number	1	2	3	4	5
Value	0	−3	−6	−9	−12

2. Starting with the number 6, build a sequence of five numbers and complete the table.

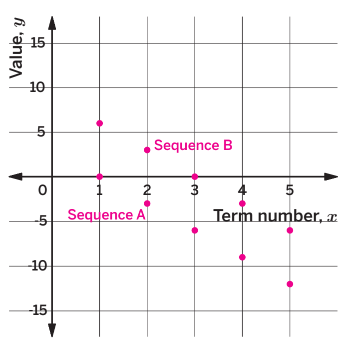
Sequence B

Term number	1	2	3	4	5
Value	6	3	0	−3	−6

3. Can you choose a starting value so that the first five terms in your sequence are all positive? Explain your thinking.

Sample response: Yes, any starting value greater than 12 generates a sequence whose first five terms are all positive.

4. Use the tables from Problems 1 and 2 to graph the two sequences on the coordinate plane. Refer to your graph for parts a and b.



a Where do you see the 3 (that is, each term was 3 less than the previous term) in your graphs?

The slope of each linear pattern is −3.

b What might a Term 0 be for each sequence? Where do you see that in your graphs?

Sequence A: Term 0 is 3 and Sequence B: Term 0 is 9 because you need to add 3 to Term 1 of each sequence to work backward. These values represent the y-intercepts of each linear pattern.

STOP

1 Launch

Conduct the *Think-Pair-Share* routine. Display the Anchor Chart PDF, *Arithmetic Sequences*, during the activity.

2 Monitor

Help students get started by annotating the table to determine the first and second values of Sequence A.

Look for points of confusion:

- **Choosing the wrong starting value in Problem 3.** Ask students whether they should increase or decrease their starting value so that all five terms are positive.
- **Connecting the points in the graph.** Ask students if it makes sense to have a point between the first and second points of each sequence in the graph.

Look for productive strategies:

- Drawing a slope triangle on the graph to determine the common difference.
- Defining a relationship between the term number and sequence value **(MP6)**.

3 Connect

Display the graph of Sequence A and Sequence B.

Have students share their responses to Problems 3–6. Select and sequence students from those who used visual strategies to those who employed more abstract strategies.

Highlight that sequences can be represented by verbal descriptions, tables, and graphs.

Ask, "What sort of functions do the graphs of your sequences remind you of?" Help students make connections with linear functions. Later in this unit, they will see how sequences and functions are related. For now, they can observe that the common difference is closely connected to the slope of a corresponding linear function.

Differentiated Support

Accessibility: *Guide Processing and Visualization, Vary Demands to Optimize Challenge*

Demonstrate for students, using colored pencils to annotate the table, how to determine the first two values in Sequence A.

Extension: *Math Enrichment*

Have students complete the following problem:

Write an equation to represent Sequence A and Sequence B from Problems 1 and 2.

Sequence A: $y = -3x + 3$

Sequence B: $y = -3x + 9$

Math Language Development

MLR7: Compare and Connect

Have pairs of students compare their tables and starting values and discuss what is the same and what is different between their tables. Have the same pair of students share and compare their graphs for Problem 4 before moving on to Problem 5.

English Learners: Display one of the tables and its corresponding graph. Add annotations to the table and graph to highlight for students the connections between the two representations.

Whole Class | 5 min

Summary

HSF.LE.A.2

Review and synthesize the process of defining arithmetic sequences and representing sequences in various ways.

Summary

In today's lesson . . .

You defined an ***arithmetic sequence*** as a sequence in which the value of each term is the value of the previous term plus a common constant. This constant can be positive, negative, or even zero. Once you know this constant, you can use it to generate other terms in the sequence.

This common constant is also called the *rate of change* or ***common difference***. Given a sequence, you can determine its common difference by subtracting any two consecutive terms in the sequence. Checking for a common difference can also help you determine whether a sequence is arithmetic. In the following arithmetic sequence, each term is 3 more than the previous term.

$$\underbrace{2 \;,\; 5}_{+3} \;,\; \underbrace{8}_{+3} \;,\; \underbrace{\_\_}_{+3} \;,\; \underbrace{\_\_}_{+3}$$

While this lesson introduced arithmetic sequences, keep in mind that there are many other types of sequences.

Reflect:

16 **Unit 1** Sequences and Series

Synthesize

Display the Anchor Chart PDF, *Arithmetic Sequences*.

Highlight that arithmetic sequences have a common difference, or rate of change, that is added to produce one term after the next. The common difference is positive when the sequence is increasing and negative when the sequence is decreasing. Arithmetic sequences can be represented by verbal descriptions, tables, and graphs.

Formalize vocabulary:

- *arithmetic sequence*
- *common difference*

Ask, "Is the following sequence arithmetic? 5, 5, 5, 5, 5 If so, how can you determine the common difference and describe the graph that represents it?" Sample response: Yes, because the common difference, $d = 0$, is the same between consecutive terms. The graph consists of the points $(1, 5)$, $(2, 5)$, $(3, 5)$. . . , which form a linear pattern.

Reflect

After synthesizing the concepts of the lesson, allow students a few moments for reflection on one of the Essential Questions for this unit. Encourage them to record any notes in the *Reflect* space provided in the Student Edition. To help them engage in meaningful reflection, consider asking:

- "How can sequences and series be represented visually?"

Math Language Development

MLR2: Collect and Display

As students formalize the new vocabulary for this lesson, ask them to refer to the class display for this unit that you started in Activity 2. Ask them to review and reflect on the terms and phrases related to the terms *common difference* and *arithmetic sequence* that were added to the display during the lesson.

Exit Ticket

HSF.LE.A.2

Students demonstrate their understanding of arithmetic sequences and the common difference.

Printable

Name: ____________ Date: ________ Period: ________

Exit Ticket 1.02

Many possible sequences can begin with the same terms.

1. Determine the next two terms of the arithmetic sequence shown, and then write the common difference.

   2, 8, **14**, **20**

   Common difference: **6**

2. Determine the next two possible terms of the sequence shown if it is *not* arithmetic. Explain your thinking.

   2, 8, **32**, **128**

   Sample response: I multiplied each term by 4 to determine the next term.

Self-Assess

1 I don't really get it | 2 I'm starting to get it | 3 I got it

a. I can explain what it means for a sequence to be arithmetic. 1 2 3

b. I can determine the common difference of an arithmetic sequence. 1 2 3

c. I can determine missing terms in an arithmetic sequence. 1 2 3

Success looks like . . .

» **Goal:** Determining the missing terms in an arithmetic sequence.

✓ Determining the next two terms in Problem 1.

» **Goal:** Determining the common difference of an arithmetic sequence.

✓ Identifying the common difference in Problem 1.

» **Language Goal:** Explaining what it means for a sequence to be arithmetic. **(Speaking)**

✓ Being able to explain why their sequence is not arithmetic in Problem 2.

Suggested next steps

If students use repeated multiplication to solve Problem 1, consider:

- Reviewing the definition of arithmetic sequences by displaying the Anchor Chart PDF, *Arithmetic Sequences*.
- Assigning Practice Problem 1.
- Asking, "What makes a sequence arithmetic?"

If students use repeated addition to solve Problem 2, consider:

- Reviewing contrasting sequences in Activity 2.
- Assigning Practice Problem 2.

Professional Learning

This professional learning moment is designed to be completed independently or collaboratively with your fellow mathematics educators. Prompts are provided so that you can reflect on this lesson before moving on to the next lesson.

Points to Ponder . . .

- What worked and didn't work today? In what ways did Activity 2: Take Turns go as planned?
- Which students' ideas were you able to highlight during Activity 2? What might you change for the next time you teach this lesson?

Name: ____ Date: ____ Period: ____

Practice

1. The first two terms of some different arithmetic sequences are shown. What are the next three terms of each sequence?

a. $-2, 4, \ldots$ **10, 16, 22**

b. $11, 111, \ldots$ **211, 311, 411**

c. $5, 7.5, \ldots$ **10, 12.5, 15**

d. $5, -4, \ldots$ **−13, −22, −31**

2. For each sequence, decide whether it could be arithmetic. Explain your thinking.

a. 

Not arithmetic; Sample response: The number of dots increases by odd numbers starting with 5.

b. 

Yes, it is arithmetic; Sample response: The number of dots increases by 3.

c.

Yes, it is arithmetic; Sample response: The number dots decreases by 2.

d. 

Not arithmetic; Sample response: The pattern is whole numbers squared starting with 2^2.

3. Complete each arithmetic sequence with its missing terms. Then state the common difference for each sequence.

a. $-3, -2,$ **−1**, **0**, 1 Common difference = **1**

b. **1**, $13, 25$, **37**, **49** Common difference = **12**

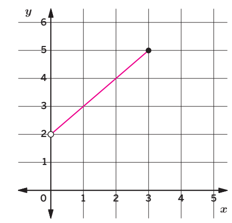
c. $1, 0.25,$ **− 0.5**, -1.25, **−2** Common difference = **−0.75**

d. $92,$ **89**, **86**, **83**, 80 Common difference = **−3**

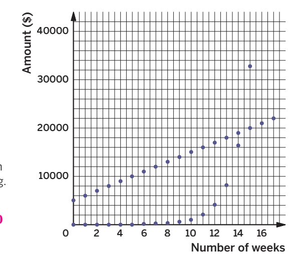
Practice

Name: ____ Date: ____ Period: ____

4. Graph the function $f(x) = x + 2$ over the domain $0 < x \leq 3$.



5. Account A starts with $5,000 and grows by $1,000 each week. Account B starts with $1 and doubles each week.



a. Which account has more money after one week? After two weeks?

Account A has more money than Account B after the first and second weeks.

b. The graph shown represents both account balances. Which graph corresponds to which account? Explain your thinking.

The linear graph corresponds to Account A because it increases with a constant rate of $1,000 a week. The nonlinear graph corresponds to Account B because it increases exponentially with a growth factor of 2.

c. Given a choice, which of the two accounts would you choose? Explain your thinking.

Sample response: I would choose Account B because after 15 weeks it accumulates more money than Account A, and the balance would continue to grow much faster.

6. Consider the sequence: $3, 6, 12, 24, 48$

a. Describe how to produce a new term from the previous term.

Multiply the previous term by 2.

b. Is the sequence arithmetic? Explain your thinking.

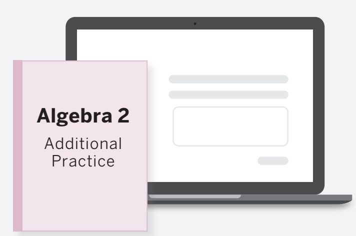
No. Sample response: Arithmetic sequences increase or decrease by adding a constant value. This sequence grows by multiplication.

Practice Problem Analysis

Type	Problem	Refer to	Standard(s)	DOK
On-lesson	1	Activity 1	HSF.LE.A.2	1
	2	Activity 2	HSF.LE.A.2	2
	3	Activity 1	HSF.LE.A.2	2
Spiral	4	Algebra 1	HSF.IF.A.1	2
	5	Algebra 1	HSF.IF.A.2	2
Formative	6	Unit 1 Lesson 3	HSF.LE.A.1	2

Power-up: If students need additional support with the key prerequisite concept or skill this problem addresses, consider assigning the Power-up in the next lesson.

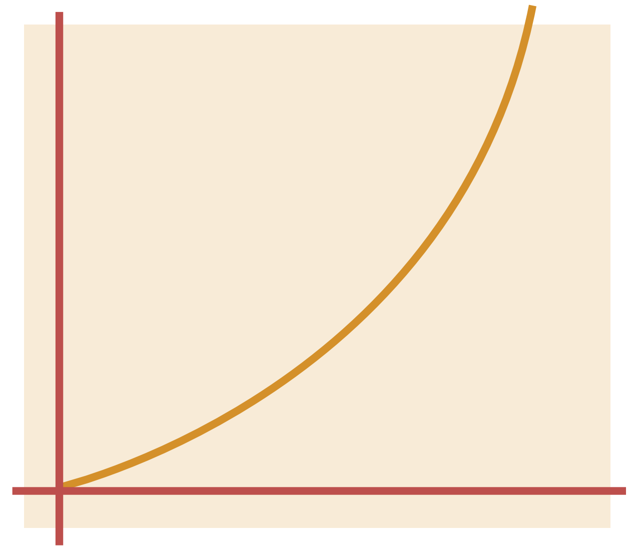
Additional Practice Available



For students who need additional practice in this lesson, assign the **Algebra 2 Additional Practice**.

Revisiting Growth and Decay

Let's explore growing and shrinking patterns.



Focus

Goals

1. Determine missing terms in a geometric sequence.
2. Determine the common ratio of a geometric sequence.
3. **Language Goal:** Explain what it means for a sequence to be geometric. **(Speaking)**

Coherence

Today

Students investigate what makes a sequence geometric. They look for and make use of structure to determine the common ratio between successive terms and determine missing terms in a geometric sequence **(MP7)**. This lesson also gives students the opportunity to use precise language to describe the relationship between consecutive terms as they formally define *geometric sequence* **(MP6)**. Students define the terms *common ratio*, *growth factor*, and *decay factor* in the context of sequences. In addition, students represent geometric patterns in real-world situations using verbal descriptions, tables, and graphs.

Previously

In Lesson 2, students identified and defined *arithmetic sequences* and recalled the terms *common difference* and *rate of change*, in the context of arithmetic sequences. In previous coursework, students studied exponential functions, and they continue to build on their understanding of exponential patterns as they make connections to geometric sequences.

Coming Soon

In Lesson 4, students will learn to write recursive rules for sequences using function notation. In an upcoming unit, students will revisit geometric sequences as they continue to explore exponential functions.

Rigor

- Students build **procedural fluency** working with geometric sequences as they determine the common ratio between terms and determine missing terms in a sequence.

Standards

Addressing

HSF.LE.A.2
Construct linear and **exponential functions, including** arithmetic and **geometric sequences, given a graph, a description of a relationship, or two input-output pairs from a table.**

Building On	Building Toward
HSF.LE.A.1.A	HSF.LE.A.2
	HSF.BF.A.2

Pacing Guide

Suggested Total Lesson Time ~**50 min**

Warm-up	Activity 1	Activity 2	Activity 3 (optional)	Summary	Exit Ticket
5 min	20 min	15 min	15 min	5 min	5 min
Independent	Pairs	Pairs	Pairs	Whole Class	Independent
MP7	MP7	MP6, MP7	MP7, MP8	MP7	
HSF.LE.A.2	HSF.LE.A.2	HSF.LE.A.2	HSF.LE.A.2	HSF.LE.A.2	HSF.LE.A.2

Amps powered by desmos | **Activity and Presentation Slides**

For a digitally interactive experience of this lesson, log in to Amplify Math at **learning.amplify.com**.

Practice Independent

Materials

- Exit Ticket
- Additional Practice
- Anchor Chart PDF, *Geometric Sequences*

Math Language Development

New words

- *common ratio*
- *fractal*
- *geometric sequence*

Review words

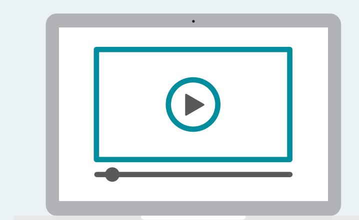
- *decay factor*
- *growth factor*

Amps | **Featured Activity**

Activity 2

Animating a Geometric Sequence

Students watch an animation of how the amount of medicine in a person's body decreases exponentially over time.



Building Math Identity and Community

Connecting to Mathematical Practices

Students might not have the organizational skills to draw a tree diagram of ancestors and relate it to the table and graph of data. Ask students to set a goal of showing their work neatly and thoroughly so that they can more easily identify the structure and patterns in a family tree **(MP7)**. This structure is what they will use to determine the values of other terms.

Modifications to Pacing

You may want to consider this additional modification if you are short on time.

- Optional **Activity 3** may be omitted.

Warm-up Which One Doesn't Belong?

MP7
HSF.LE.A.2

Students determine which sequence does not belong to prepare for a more formal approach to analyzing geometric sequences **(MP7).**

Name: Date: Period:

Unit 1 | Lesson 3

Revisiting Growth and Decay

Let's explore growing and shrinking patterns.

Warm-up Which One Doesn't Belong?

Which of the following sequences does not belong with the others? Explain your thinking.

Sequence A: $32, 24, 16, 8, \ldots$

Sequence B: $1, 4, 16, 64, \ldots$

Sequence C: $4, 4, 4, 4, \ldots$

Sequence D: $8, 12, 16, 20, \ldots$

Sample responses:

- Sequence A does not belong because it is the only sequence that is decreasing.
- Sequence B does not belong because it is the only sequence that includes an odd number.
- Sequence C does not belong because it is the only sequence that does not change.
- Sequence D does not belong because it is the only sequence that increases by a constant amount.

Log in to Amplify Math to complete this lesson online.

1 Launch

Conduct the ***Which One Doesn't Belong?*** routine and remind students there are no incorrect answers.

2 Monitor

Help students get started by having them compare two sequences at a time and determine how they are changing.

Look for productive strategies:

- Annotating each sequence to note differences or similarities.
- Creating other representations of each sequence, such as a table or a graph, to model and observe the patterns and structure of the sequence. For example, Sequence A can be modeled by a linear graph, while Sequence B can be modeled by an exponential graph.

3 Connect

Have students share one reason for why a sequence does not belong with the others. After each response, ask the class whether they agree or disagree.

Display all four sequences.

Highlight that there is no single correct answer, and focus on the reasoning and explanations. Direct attention toward Sequence B, which is growing in a different way from the other sequences — specifically, each term increases by a factor of 4.

Math Language Development

MLR2: Collect and Display

Collect informal and formal language students use to describe the reasons why each sequence might not belong with the others. Add these terms and phrases to the class display and refer to it throughout the remainder of the lesson as students continue to make sense of different types of sequences.

Power-up

To power up students' ability to express linear and exponential functions appropriately, ask:

1. Is the pattern linear or exponential? Explain your thinking.

a

x	0	1	2
y	3	6	9

Linear because the rate of change is 3.

b

x	0	1	2
y	3	6	12

Exponential because the growth factor is 2.

2. Write an equation for each pattern.

a $y = 3 + 3x$ b $y = 3(2^x)$

Use: Before the Warm-up

Informed by: Performance on Lesson 2, Practice Problem 6 and the Pre-Unit Readiness Assessment, Problem 6

Pairs | 20 min

MP7
HSF.LE.A.2

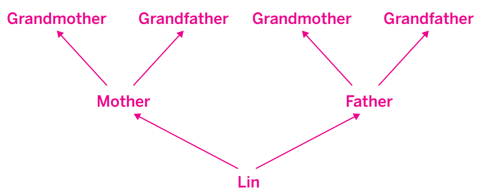
Activity 1 Family Tree

Students explore an increasing exponential pattern in a real-world problem and make connections to properties of geometric sequences with tables and graphs.

Activity 1 Family Tree

Let's explore mathematical patterns in ancestry.

1. Suppose Lin is tracing the genealogy of her family tree. She knows that she has two parents in the first generation, four grandparents in the second generation, and so on. Draw a family tree for Lin that represents her direct ancestors.



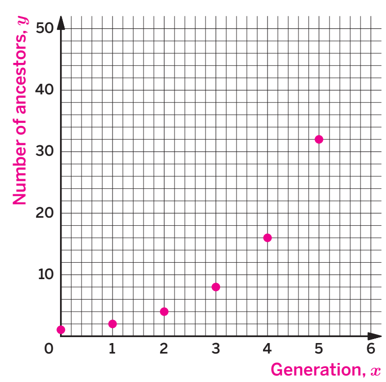
2. If Lin represents Generation 0, how many direct ancestors does she have in each previous generation? Complete the table to illustrate Lin's ancestry from the first to the fifth generations.

Generation	0	1	2	3	4	5
Number of ancestors	1	2	4	8	16	32

3. Graph the number of direct ancestors as a function of the generation.

   a. List the number of ancestors as a sequence. What is the ratio between successive terms?
   1, 2, 4, 8, 16, 32. The ratio is 2.

   b. Where do you see this ratio in the graph?
   The y-values are doubling for every x-value. This pattern looks like an exponential one that is growing by a factor of 2.



20 Unit 1 Sequences and Series

1 Launch

Activate students' background knowledge by asking, "Have you ever seen or created a family tree? What does it mean to be an ancestor of someone, rather than just a relative?" Sample response: A direct *ancestor* is someone from whom you have descended, such as a parent or grandparent, while a *relative* could be anyone in the same family connected by blood, marriage, or adoption. Emphasize that for the purposes of this activity, students will consider direct ancestors as they explore patterns in genealogy. Conduct the ***Think-Pair-Share*** routine.

2 Monitor

Help students get started by modeling how to draw a tree diagram.

Look for points of confusion:

- **Incorrectly determining the number of ancestors.** Annotate the tree diagram by labeling Generation 1 and Generation 2. Ask, "How does the number of ancestors change from one generation to the next?"
- **Connecting the points in the graph in Problem 3.** Ask students if it makes sense to have a generation between the first and second.

Look for productive strategies:

- Using structure to analyze the pattern and determine the third, fourth, and fifth terms **(MP7)**.
- Extending the table to explore Problem 4.

Activity 1 continued ›

Differentiated Support

Accessibility: *Clarify Vocabulary and Symbols*

During the Connect, refer students to the class display from the Warm-up activity. Add the vocabulary terms *common ratio* and *geometric sequence*, along with their definitions, to the class display. Demonstrate where to see the common ratio in a geometric sequence by adding the sequence from Problem 2 to the display and annotating the common ratio between successive terms.

Extension: *Math Enrichment*

Have students generalize a rule for calculating their solution to Problem 3.

Sample response: $y = 2^x$

Math Language Development

MLR1: Stronger and Clearer Each Time

After students complete Problems 2 and 3, have pairs meet with 1–2 other pairs of students to share their responses. Encourage reviewers to ask clarifying questions such as:

- "What pattern did you notice in the table?"
- "How did you identify the common ratio?"
- "Can you include a diagram to support your explanation?"

Have students revise their responses, as needed.

English Learners: Use structured pairing to pair students together with varying levels of English language proficiency. This will give students an opportunity to engage with a diverse group of peers to support each other with the linguistic demands of the routine.

Pairs | 20 min

Activity 1 Family Tree (continued)

MP7
HSF.LE.A.2

Students explore an increasing exponential pattern in a real-world problem and make connections to properties of geometric sequences with tables and graphs.

Name: ______ Date: ______ Period: ______

Activity 1 Family Tree (continued)

4. How many direct ancestors did Lin have 10 generations ago? 1,024

5. Determine the missing values in each geometric sequence and the corresponding common ratio.

1, 3, 9, 27, 81 Common ratio = 3

2, 5, 12.5, 31.25, 78.125 Common ratio = 2.5

0.5, 2, 8, 32, 128 Common ratio = 4

3 Connect

Display the Anchor Chart PDF, *Geometric Sequences*, and Problems 2 and 3.

Have pairs of students share their responses to Problems 2 and 3 and annotate the geometric sequence in the same format shown in the Anchor Chart PDF.

Highlight that students can determine whether a sequence is geometric by calculating the *common ratio* between successive terms, and identifying whether it is constant.

Define ***geometric sequence*** as a sequence in which each term is the previous term multiplied by a constant, known as the ***common ratio***.

Pairs | 15 min

Activity 2 Less and Less

MP6, MP7
HSF.LE.A.2

Students explore a decreasing exponential pattern in a real-world problem and make connections between properties of geometric sequences and tables and graphs.

Amps Featured Activity Animating a Geometric Sequence

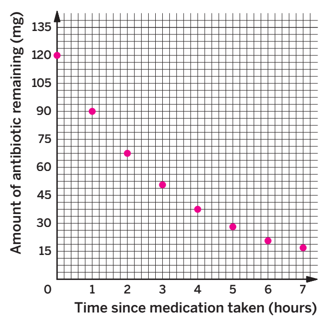
Activity 2 Less and Less

The amount of medicine in a person's bloodstream decreases over time in a way that can be described mathematically. Priya takes 120 mg of an antibiotic every 8 hours for a week. After 1 hour, 75% of that amount, or 90 mg, remains in her bloodstream.

1. Complete the table by recording how much antibiotic is in Priya's bloodstream every hour. Round to the nearest tenth.

Time since medication taken (hours)	0	1	2	3	4	5	6	7
Amount of antibiotic (mg)	120	90	67.5	50.6	38	28.5	21.4	16

2. Graph the amount of antibiotic remaining in Priya's system as a function of time.



3. What pattern do you notice in the graph?
The amount of medicine is decreasing in a way that looks exponential.

4. Determine the ratio between successive terms, and then explain what this means in context.
The ratio between successive terms is 0.75, so 75% of the amount of medicine from the previous hour remains in Priya's bloodstream.

1 Launch

Read the prompt as a class. Conduct the ***Think-Pair-Share*** routine. Display the Anchor Chart PDF, *Geometric Sequences*, during the activity.

2 Monitor

Help students get started by completing the first two columns of the table.

Look for points of confusion:

- **Incorrectly calculating the common ratio.** Ask, "How do you determine the common ratio between consecutive terms?" Encourage students to describe the process using mathematical language **(MP6)**.
- **Thinking the common ratio is larger than 1.** Ask, "Will a decreasing geometric sequence have a common ratio less than 1 or greater than 1?"

Look for productive strategies:

- Determining the decay factor between successive terms.
- Using structure to analyze the patterns to determine the missing terms **(MP7)**.

Activity 2 continued ›

Differentiated Support

Accessibility: *Optimize Access to Technology*

Have students use the Amps slides for this activity, in which they watch an animation of how the amount of medicine in a person's body decreases exponentially over time.

Math Language Development

MLR7: Compare and Connect

After students complete Problem 3, have pairs discuss the similarities and differences in the patterns they recognize in the graph. Encourage students to discuss patterns in context such as, "The amount of medicine in a person's bloodstream decreases exponentially each hour."

English Learners: As students discuss similarities and differences, circulate and listen to student conversations. Highlight language students use and connect their language to those that have been added to the class display. This will support students' sensemaking with their developing mathematical language around exponential relationships.

Pairs | 15 min

Activity 2 Less and Less (continued)

MP6, MP7
HSF.LE.A.2

Students explore a decreasing exponential pattern in a real-world problem and make connections between properties of geometric sequences and tables and graphs.

Name: ______ Date: ______ Period: ______

Activity 2 Less and Less (continued)

5. The rate at which a person's body absorbs medicine depends on several factors, such as a person's body weight. The following sequences represent the amount of medicine remaining in the bloodstream of three different people as a function of time. Determine the missing values in each geometric sequence and the corresponding common ratio. Round to the nearest tenth.

a. **200**, 160, 128, 102.4, **81.9** Common ratio = **0.8**

b. 300, 270, **243**, **281.7**, 196.8 Common ratio = **0.9**

c. 240, **120**, 60, 30, **15** Common ratio = **0.5**

Are you ready for more?

Determine the next three terms in the following sequence: $-625, 125, -25, 5, \ldots$

$-1, \frac{1}{5}, -\frac{1}{25}$

3 Connect

Display the table and graph.

Have pairs of students share their conclusions and strategies for determining the values in the table and the common ratio. Select and sequence pairs to share their responses.

Highlight that geometric sequences increase or decrease by a *common ratio* with each term. If the common ratio is greater than 1, then the sequence increases. If the common ratio is between 0 and 1, then the sequence decreases.

Ask:

- "If you were only given the values of the 2nd and 4th terms of a sequence, how could you determine the common ratio and calculate missing terms?"
- "For example, what are the missing values in this sequence? ____, 54, ____, 6, ____"
  162, 18, 2; Sample response: Divide 6 by 54 and take the square root.
  $\frac{6}{54} = \frac{1}{9}$ and $\sqrt{\frac{1}{9}} = \frac{1}{3}$. The common ratio is $\frac{1}{3}$.

Optional

Pairs | 15 min

MP7, MP8
HSF.LE.A.2

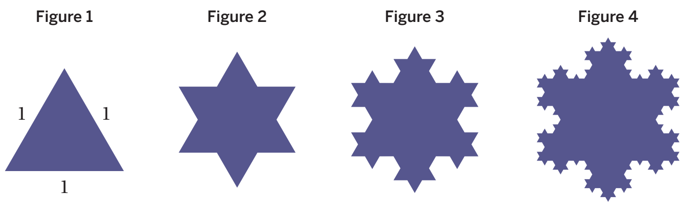
Activity 3 The Koch Snowflake

Students calculate the perimeter of successive figures to better understand the recursive nature of fractals and geometric sequences.

Activity 3 The Koch Snowflake

Patterns that repeat as you zoom in or out, or *fractals*, can be found in the world all around us, from snowflakes to mountain ranges. Hee Oh, a Yale professor and mathematician, studies fractals and their applications to fields such as medicine, climate science, and the efficient transfer of digital information. Fractal patterns can also be created by math equations based on simple rules. Investigate how the perimeter of a simple shape can grow more and more complex.

Figure 1 is an equilateral triangle with side lengths of 1 unit. Smaller equilateral triangles are added to every side of each figure to obtain the next figure in the sequence.



1. What is the perimeter of Figure 1? **3 units**

2. What is the perimeter of Figure 2? Show your thinking. **4 units; $3\left(\frac{1}{3}+\frac{1}{3}+\frac{1}{3}+\frac{1}{3}\right)=4$**

3. Continue the pattern and complete the following table.

Figure number	1	2	3	4	5
Perimeter (units)	3	4	$\frac{16}{3}$	$\frac{64}{9}$	$\frac{256}{27}$

1 Launch

Display the diagram of figures and activate background knowledge by asking, "Have you ever closely examined a snowflake or frosted glass?" Conduct the ***Think-Pair-Share*** routine.

2 Monitor

Help students get started by reminding them that the side lengths are 1 unit in Figure 1.

Look for points of confusion:

- **Misinterpreting how the triangle is changing.** Explain that the 1st figure is an equilateral triangle and that the length of each side for any figure is $\frac{1}{3}$ the length of a side from the previous figure **(MP7)**.

Look for productive strategies:

- Looking for a common difference or common ratio.

Activity 3 continued ›

Math Language Development

MLR8: Discussion Supports — Press For Reasoning

Use this routine as students complete Problem 4. Press students for details in their reasoning by asking, "How do you know the relationship is exponential?," and encourage students to make connections to their work in Problem 3.

English Learners: Encourage students to refer to and use language from the class display to support their reasoning about the exponential relationship.

Pairs | 15 min

MP7, MP8
HSF.LE.A.2

Activity 3 The Koch Snowflake (continued)

Students calculate the perimeter of successive figures to better understand the recursive nature of fractals and geometric sequences.

Name: Date: Period:

Activity 3 The Koch Snowflake (continued)

4. Does the pattern grow linearly or exponentially? If linearly, what is the rate of change? If exponentially, what is the growth or decay factor?
The pattern grows exponentially because the perimeter increases by a growth factor of $\frac{4}{3}$ with each figure.

5. How might you determine the perimeter of Figure n?
Sample response: Figure 1 has a perimeter of 3. Figure 2's perimeter is calculated by multiplying 3 by $\frac{4}{3}$. Figure 3's perimeter is calculated by multiplying Figure 2's perimeter by $\frac{4}{3}$. I could determine the perimeter of Figure n by multiplying the perimeter of Figure $n-1$ by $\frac{4}{3}$, or by multiplying 3 by $\left(\frac{4}{3}\right)^{n-1}$.

Featured Mathematician

Hee Oh

Hee Oh is a South Korean mathematician and the first female tenured Professor of Mathematics at Yale University. She works in a branch of mathematics called *chaos theory*, which focuses on fractal patterns, such as the Koch snowflake. Her research has wide-ranging applications, including interpreting heartbeat irregularities, weather and climate change, pandemic crisis management, and other complex natural systems that appear random and disordered, yet have inherent structure.

Hee Oh STOP

3 Connect

Display a digital animation of the figures approaching the Koch snowflake.

Have students share their strategies for determining the perimeters of the figures. Select and sequence students to share their thinking.

Highlight that the perimeters of the figures form a geometric sequence with a common ratio of $\frac{4}{3}$. To calculate this ratio, divide the perimeter of any figure by the perimeter of the previous figure. If necessary, recall that dividing by a fraction is the same as multiplying by its reciprocal.

Define ***fractal*** as a geometric figure with a repeating pattern that appears the same at different scales.

Ask, "Do you think the snowflake has a finite perimeter, or does it grow larger and larger without bound?" **(MP8)** Sample response: It grows larger forever.

Featured Mathematician

Hee Oh

Hee Oh is a South Korean mathematician who works in a branch of mathematics called Chaos Theory, which focuses on fractal patterns in natural systems.

Whole Class | 5 min

Summary

MP7
HSF.LE.A.2

Review and synthesize the process of defining geometric sequences, representing sequences in various ways, and relating sequences with exponential functions.

Summary

In today's lesson . . .

You defined a ***geometric sequence*** as a sequence in which the value of each term is the value of the previous term multiplied by a constant, called a ***common ratio***. The common ratio can be found by dividing any term by the preceding term. Checking for a common ratio can also help you determine whether a sequence is geometric.

For example, in the following geometric sequence, each term is 3 times the term before it.

$$2 \underset{\cdot 3}{\longrightarrow} 6 \underset{\cdot 3}{\longrightarrow} 18 \underset{\cdot 3}{\longrightarrow} 54 \underset{\cdot 3}{\longrightarrow} 162$$

One application of geometric sequences is the formation of *fractals*. A ***fractal*** is a pattern that keeps repeating itself as you zoom in or out. Fractals are useful for modeling coastlines, galaxies, blood vessels, crystals, algae, snowflakes, and many other things in nature. Cell phone antennas are more efficient because they are based on fractals.

Reflect:

26 **Unit 1** Sequences and Series

Synthesize

Display the Anchor Chart PDF, *Geometric Sequences*.

Highlight that geometric sequences have a common ratio or growth factor that is multiplied to produce one term after the next. This differs from arithmetic sequences, where a common difference or rate of change is added to produce one term after the next **(MP7)**. Both arithmetic and geometric sequences can be represented by verbal descriptions, tables, and graphs.

Formalize vocabulary:

- *common ratio*
- *fractal*
- *geometric sequence*

Ask, "How do geometric sequences compare to the arithmetic sequences that you studied in the previous lesson?"

Reflect

After synthesizing the concepts of the lesson, allow students a few moments for reflection on one of the Essential Questions for this unit. Encourage them to record any notes in the *Reflect* space provided in the Student Edition. To help them engage in meaningful reflection, consider asking:

- "How can simple rules lead to complex patterns in sequences and series?"

Math Language Development

MLR2: Collect and Display

As students formalize the new vocabulary for this lesson, ask them to refer to the class display for this unit that you started in Lesson 2. Ask them to review and reflect on the terms and phrases related to the terms *common ratio, fractal, geometric sequence,* and *common ratio* that were added to the display during the lesson.

Independent | 5 min

Exit Ticket

HSF.LE.A.2

Students demonstrate their understanding of what defines a geometric sequence and the common ratio.

Printable

Name: ______ Date: ______ Period: ______

Exit Ticket 1.03

Consider the sequence 500, 100, 20, . . .

1. Explain why this sequence could represent a geometric sequence.
Sample response: A previous term is multiplied by $\frac{1}{5}$ to produce the next term, so this could be a geometric sequence with a common ratio of $\frac{1}{5}$.

2. What is the next term in the sequence? Explain your thinking.
4; Sample response: Multiply the previous term by $\frac{1}{5}$.

Self-Assess

1 I don't really get it | 2 I'm starting to get it | 3 I got it

a. I can explain what it means for a sequence to be geometric.
1 2 3

b. I can determine the common ratio of a geometric sequence.
1 2 3

c. I can determine missing terms in a geometric sequence.
1 2 3

Success looks like . . .

» **Goal:** Determining missing terms in a geometric sequence.

✓ Determining the correct term in Problem 2.

» **Goal:** Determining the common ratio of a geometric sequence.

✓ Identifying the common ratio in their explanations in Problem 1 or 2.

» **Language Goal:** Explaining what it means for a sequence to be geometric. **(Speaking)**

✓ Explaining why the represented sequence is geometric in Problem 1.

Suggested next steps

If students cannot determine if the sequence is geometric in Problem 1, consider:

- Reviewing the definition of geometric sequences by displaying the Anchor Chart PDF, *Geometric Sequences*.
- Reviewing Activity 2.
- Assigning Practice Problem 3.
- Asking, "What makes a sequence geometric?"

If students are unable to determine the next term of the sequence in Problem 2, consider:

- Reviewing how to calculate the common ratio.
- Assigning Practice Problem 2.

Professional Learning

This professional learning moment is designed to be completed independently or collaboratively with your fellow mathematics educators. Prompts are provided so that you can reflect on this lesson before moving on to the next lesson.

Points to Ponder . . .

- What worked and didn't work today? What surprised you as your students worked on Activity 2?
- How was today's lesson on geometric sequences similar to or different from the previous lesson on arithmetic sequences? What might you change for the next time you teach this lesson?

Math Language Development

MLR1: Stronger and Clearer Each Time

In Activity 1, you used structured pairing with MLR1 to group students with varying levels of English language proficiencies. Use these prompts to reflect on this routine.

- What effect did this grouping strategy have on partner discussions, written explanations and peer feedback?
- Would you change anything the next time you use MLR1?

Practice

Independent

Name: ______ Date: ______ Period: ______

Practice

1. For each sequence, decide whether it is *arithmetic* or *geometric*.

a. 2, 4, 16, . . .
geometric

b. 6, 12, 18, . . .
arithmetic

c. 10, 20, 30, . . .
arithmetic

d. 100, 20, 4, . . .
geometric

e. 200, 40, 8, . . .
geometric

2. The first two terms of a sequence are 1 and 10.

a. Determine the next two terms if the sequence is arithmetic.
19, 28

b. Determine the next two terms if the sequence is geometric.
100, 1000

c. What are the next possible terms if the sequence is neither arithmetic or geometric? Describe your sequence.
Answers may vary, provided they are neither arithmetic nor geometric. Sample response: 1, 10, 1, 10, . . . The sequence cycles between 1 and 10.

3. Complete each geometric sequence by determining the missing terms. Then state the common ratio for each sequence.

a. 1, 5, 25, 125, 625 — Common ratio = 5

b. −1, 6, −36, 216, −1296 — Common ratio = −6

c. 10, 5, 2.5, 1.25, 0.625 — Common ratio = 0.5

d. 4, −12, 36, −108, 324 — Common ratio = −3

e. 8, 12, 18, 27, 40.5 — Common ratio = 1.5

Practice

Name: ______ Date: ______ Period: ______

4. Diego purchased a car for $25,000. His car depreciates in value by 10% each year.

a. Write an exponential function, C, that models the car's value t years after purchase.
$C(t) = 25000(0.90)^t$

b. How much is Diego's car worth after 5 years?
$C(5) = 25000(0.90)^5 = \$14{,}762.25$

5. The height in inches of water in a bathtub, w, is a function of time, t, in minutes. Let P represent this function. Match each expression with its corresponding function notation.

a. After 20 minutes, the bathtub is empty. — c $P(10) = 4$

b. The bathtub is initially without water. — e $P(t) = w$

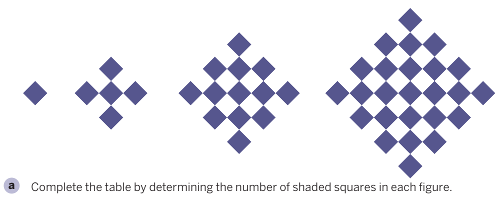
c. After 10 minutes, the height of the water is 4 in. — a $P(20) = 0$

d. The height of the water is 10 in. after 4 minutes. — b $P(0) = 0$

e. The height of the water is w in. after t minutes. — d $P(4) = 10$

6. Study the pattern shown.

Figure 1 Figure 2 Figure 3 Figure 4



a. Complete the table by determining the number of shaded squares in each figure.

Figure number	1	2	3	4	5	6
Number of shaded squares	1	5	13	25	41	61

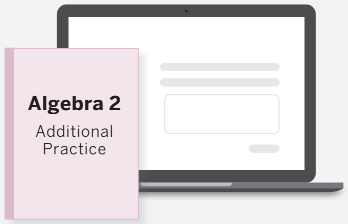
b. If n represents the figure number, and $C(n)$ represents the number of shaded squares, how many shaded squares are in $C(5)$ and $C(6)$?
$C(5) = 41$ and $C(6) = 61$

Practice Problem Analysis

Type	Problem	Refer to	Standard(s)	DOK
On-lesson	1	Activity 1	HSF.LE.A.2	1
	2	Activity 1	HSF.LE.A.2	2
	3	Activity 2	HSF.LE.A.2	2
Spiral	4	Algebra 1	HSF.IF.A.1	2
	5	Algebra 1	HSF.IF.A.2	2
Formative	6	Unit 1 Lesson 4	HSF.LE.A.1	2

Power-up: If students need additional support with the key prerequisite concept or skill this problem addresses, consider assigning the Power-up in the next lesson.

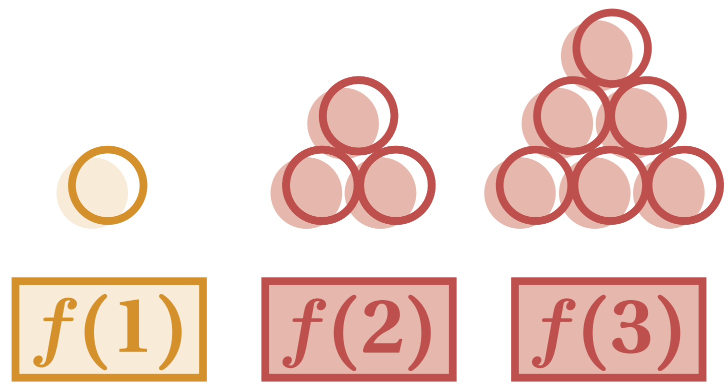
Additional Practice Available



For students who need additional practice in this lesson, assign the **Algebra 2 Additional Practice**.

Sequences Are Functions

Let's learn how to define a sequence recursively.



Focus

Goals

1. Comprehend that sequences are functions whose domain is a subset of the integers.
2. Create a recursive rule for a sequence using function notation.

Coherence

Today

Students recall the concept of functions and function notation from algebra and learn that sequences are functions with a restricted integer domain. They practice writing recursive rules for different types of sequences in function notation using repeated reasoning **(MP8)**. In addition, students build on previous informal language to make connections between arithmetic and geometric sequences and linear and exponential functions.

Previously

In prior years, students studied linear and exponential functions, as well as domain. In Lessons 2 and 3, students analyzed and defined arithmetic and geometric sequences using common differences and common ratios.

Coming Soon

In future lessons, students will write explicit rules for the nth term of both arithmetic and geometric sequences, in mathematical and real-world situations.

Rigor

- Students develop **conceptual understanding** of sequences by recognizing that they are functions whose domain is a subset of the integers.
- Students develop **procedural fluency** recursively defining sequences with an equation by using function notation.

Standards

Addressing

HSF.BF.A.2
Write arithmetic and geometric sequences both **recursively** and with an explicit formula, use them **to model situations**, and translate between the two forms.

Also Addressing: **HSF.IF.A.3, HSF.LE.A.2**

Building On	Building Toward
HSF.IF.A	HSF.BF.A.2
	HSF.IF.A.3

Pacing Guide

Suggested Total Lesson Time ~**50 min**

Warm-up	Activity 1	Activity 2	Activity 3	Summary	Exit Ticket
5 min	15 min	10 min	10 min	5 min	5 min
Independent	Pairs	Pairs	Pairs	Whole Class	Independent
	MP8	MP7	MP7, MP8		
HSF.LE.A.2	HSF.BF.A.2, HSF.IF.A.3	HSF.BF.A.2, HSF.IF.A.3	HSF.BF.A.2, HSF.IF.A.3	HSF.BF.A.2, HSF.IF.A.3	HSF.BF.A.2

Amps powered by desmos | Activity and Presentation Slides

For a digitally interactive experience of this lesson, log in to Amplify Math at **learning.amplify.com**.

Practice Independent

Materials

- Exit Ticket
- Additional Practice
- Activity 2 PDF, pre-cut cards, one per pair
- Anchor Chart PDF, *Function Notation* (from Algebra 1)
- Anchor Chart PDF, *Recursive Rules*

Math Language Development

New words

- ***recursive rule***
- ***triangular numbers***

Review words

- *arithmetic sequence*
- *decay factor*
- *domain*
- *exponential function*
- *function notation*
- *geometric sequence*
- *growth factor*
- *linear function*
- *rate of change*

Amps | Featured Activity

Activity 3
Using Work From Previous Slides

Students build on their work from previous slides to help them write recursive rules.

Amps POWERED BY desmos

Building Math Identity and Community

Connecting to Mathematical Practices

Students might feel overwhelmed by the task of sorting the cards and their emotional state might cause them to act impulsively, trying to finish quickly rather than sort the cards correctly. Ask students to work with their partner to set up a system, or structure, for how they will sort the cards **(MP7)**. For each card, they need to identify the options and how they will determine the correct way to sort it. Then have the partners encourage each other to work within that structure in order to prevent nonproductive emotional responses.

Modifications to Pacing

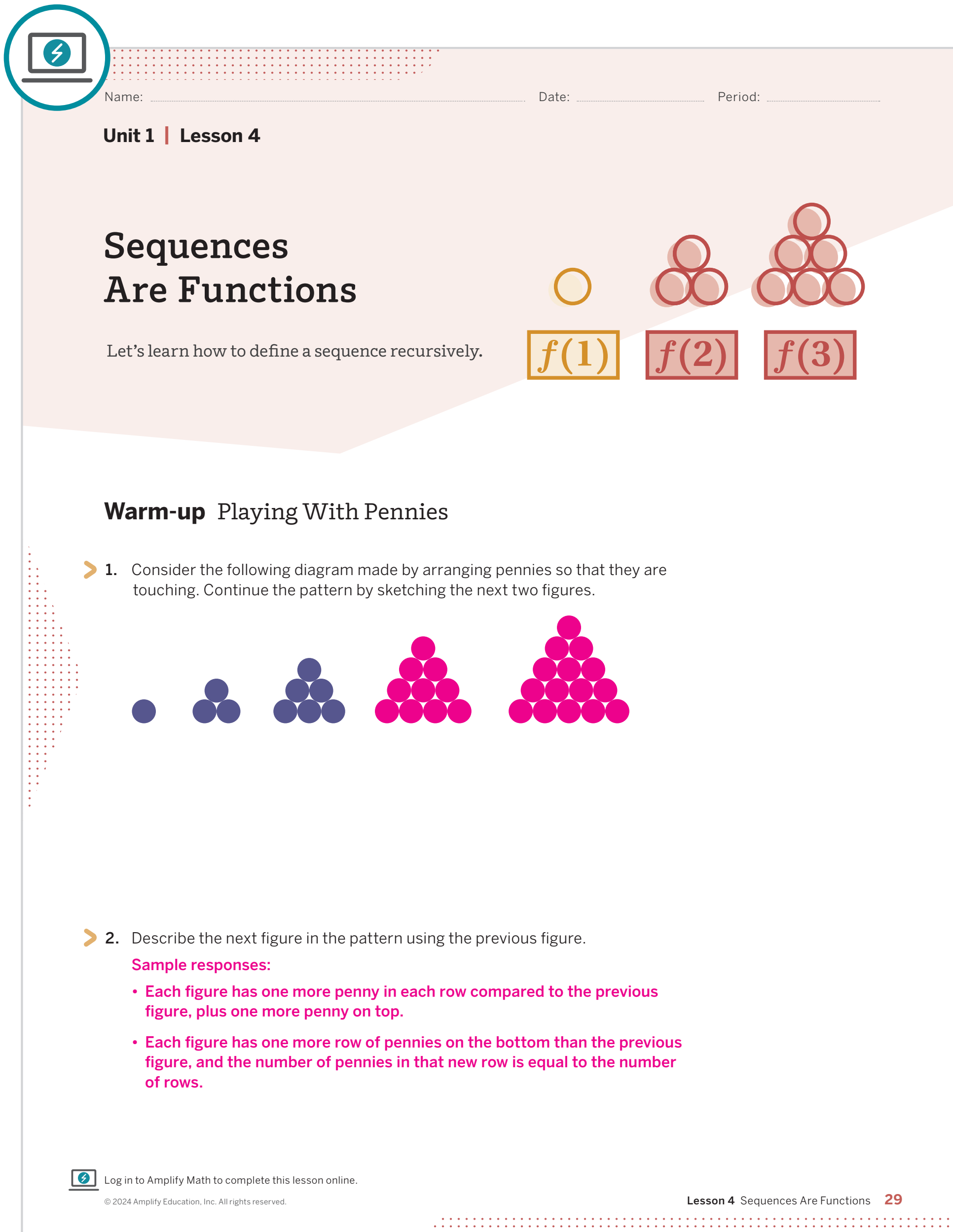
You may want to consider these additional modifications if you are short on time.

- In **Activity 1**, omit the Historical Moment.
- Reduce the number of cards in **Activity 2**.
- Omit Sequences C and E in **Activity 3**.

Warm-up Playing With Pennies

HSF.LE.A.2

Students extend a visual pattern and use informal language to describe successive figures to prepare them for writing recursive rules.



Name: Date: Period:

Unit 1 | Lesson 4

Sequences Are Functions

Let's learn how to define a sequence recursively.

$f(1)$ $f(2)$ $f(3)$

Warm-up Playing With Pennies

1. Consider the following diagram made by arranging pennies so that they are touching. Continue the pattern by sketching the next two figures.

2. Describe the next figure in the pattern using the previous figure.

   Sample responses:

   - Each figure has one more penny in each row compared to the previous figure, plus one more penny on top.
   - Each figure has one more row of pennies on the bottom than the previous figure, and the number of pennies in that new row is equal to the number of rows.

Log in to Amplify Math to complete this lesson online.

© 2024 Amplify Education, Inc. All rights reserved.

Lesson 4 Sequences Are Functions 29

1 Launch

Conduct the ***Think-Pair-Share*** routine.

2 Monitor

Help students get started by prompting them to label the total number of pennies in each figure while looking for a pattern.

Look for points of confusion:

- **Describing the pattern using the term number rather than the previous figure.** Ask students, "What is changing from one figure to the next? How do you see it growing?"

Look for productive strategies:

- Annotating the pattern to show how it is changing.
- Using words, such as *next* and *previous*, to describe the relationship between figures.

3 Connect

Display the pattern.

Have students share their responses. Record their responses on the diagram as a reference for the next activity.

Highlight that the number of pennies in each figure is increasing and that they will learn how to describe the pattern using a rule.

Ask, "If this sequence is called a function, T, then what does T represent? What is the value of $T(4)$? $T(5)$?" Sample response: T represents the total number of pennies in each figure. $T(4) = 10$ and $T(5) = 15$.

Math Language Development

MLR7: Compare and Connect

Ask pairs of students to compare the different ways they visualize the pattern growing. Encourage students to identify and share similarities and differences in the way they visualize the pattern's growth.

Power-up

To power up students' ability to use function notation to determine values of a function, ask:

Use the function $f(x) = 4 \cdot 3^x$ to determine the value of each expression.

a. $f(0)$ $= 4$

b. $f(1)$ $= 12$

c. $f(2)$ $= 36$

d. $f(3)$ $= 108$

Use: Before the Warm-up

Informed by: Performance on Lesson 3, Practice Problem 6 and the Pre-Unit Readiness Assessment, Problem 2

Activity 1 How Does It Grow?

MP8
HSF.BF.A.2, HSF.IF.A.3

Students formalize that sequences are functions with integer inputs and write their own recursive rules for sequences using function notation.

Activity 1 How Does It Grow?

Consider the arrangements of pennies from the Warm-up.

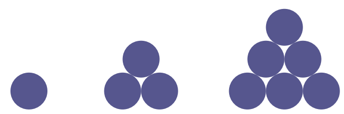


Figure 1 Figure 2 Figure 3

1. The total number of pennies in each figure can be considered a function, T, of the figure number, or term number, n.

   a. Complete the table.

Term number, n	Number of pennies, $T(n)$
1	$T(1) = 1$
2	$T(2) = 3$
3	$T(3) = 6$
4	$T(4) = 10$
5	$T(5) = 15$

   b. Define each value of the function in terms of the previous value. For example, $T(2) = T(1) + 2$. Write an expression for $T(3)$ in terms of $T(2)$ and an expression for $T(4)$ in terms of $T(3)$.
   $T(3) = T(2) + 3$ and $T(4) = T(3) + 4$.

   c. Describe how to determine the number of pennies in each term by completing the following statement:
   "The number of pennies in the current term is equal to the number of pennies in the previous term plus the term number."

30 Unit 1 Sequences and Series

1 Launch

Display the pattern from the Warm-up, including student comments. Activate prior knowledge about functions and emphasize how they can often be represented by a verbal description, table, graph, or equation. Display the Anchor Chart PDF, *Function Notation* for the remainder of the lesson.

2 Monitor

Help students get started by annotating the first two rows of the table in Problem 1 in the same format as the Anchor Chart PDF, *Function Notation*.

Look for points of confusion:

- **Trying to define the pattern as arithmetic or geometric.** Ask students what it means for a sequence to be arithmetic or geometric, and to then recheck the sequence.
- **Labeling the previous term with an incorrect expression.** Ask, "If n represents the position of the current term, how can you express the position of the previous term? For example, if $n = 3$, then how could you represent the previous term?"
- **Not understanding that the domain is restricted to positive integer values.** Ask if it makes sense for the term number n to represent -3, 0.5, or $\sqrt{2}$.

Look for productive strategies:

- Annotating the figure and the table using function notation.
- Using repeated reasoning to write a recursive rule for function T **(MP8)**.

Activity 1 continued >

Differentiated Support

Accessibility: *Activate Prior Knowledge*

Remind students they learned about functions in previous grades by displaying the Anchor Chart PDF, *Function Notation* to highlight how functions can be represented in multiple ways. This will help students during the Connect section when they make sense of the fact that sequences are functions with integer domains.

Accessibility: *Guide Processing and Visualization*

Demonstrate and encourage students to use color coding and annotations to highlight connections between the pattern, verbal description, table, and recursive equation.

Math Language Development

MLR8: Discussion Supports

Use this routine as students complete Problem 5. Have students meet with 1–2 peers to review and give feedback. Provide example prompts such as:

- "Why do you think 1.5 will not work?"
- "Do you think a number less than 0 makes sense?"

English Learners: Before students meet with their first partner, demonstrate by role playing with a student volunteer how you would provide feedback using one of the example prompts.

Activity 1 How Does It Grow? (continued)

MP8
HSF.BF.A.2, HSF.IF.A.3

Students formalize that sequences are functions with integer inputs and write their own recursive rules for sequences using function notation.

Name: ______ Date: ______ Period: ______

Activity 1 How Does It Grow? (continued)

2. Write an equation for $T(n)$ in terms of the previous values of the function. Explain your thinking.
   Sample response: The current term = previous term + n and can be modeled by function notation: $T(n) = T(n-1) + n$.

3. What values of n make sense for this equation? Explain your thinking.
   Sample response: Because n is the term number, numbers such as 2, 3, and 4 make sense.

4. Is it reasonable to include $n = 1$ in the recursive rule of $T(n)$? Why or why not?
   Sample response: No, because if you substitute $n = 1$ into the equation, $T(0)$ is not defined.

5. What values of n do not make sense? Explain your thinking.
   Sample response: Because n is the term number, numbers such as 1.5 or 2.5 do not make sense because you cannot draw a row with a partial number of dots. Also, numbers less than or equal to 0 do not make sense.

6. What is the domain of the function, T?
   The domain is all the possible term numbers represented by the variable n or $n = 1, 2, 3, 4, \ldots$

Historical Moment

Ancient Greek mathematicians were particularly interested in numbers that correspond to geometric figures because they believed these patterns reflected the inherent beauty and symmetry of the universe. The number sequence formed by the pennies is called the triangular number sequence because each term can be arranged into a triangle.

Can you think of any other shapes that can be formed from number patterns?
Sample responses:
rectangles squares

3 Connect

Display the table from Problem 1.

Have pairs of students share their responses and reasoning for Problems 1 and 2. Record their expressions by extending the table.

Highlight that sequences are functions with integer domains. A function can be represented with a verbal description, table, or recursive equation. Students will continue to explore other representations in the next lesson. When working with functions, it is important to define a domain that makes sense given the context.

Define ***recursive rule*** as a formula where a term is defined by its preceding term and ***triangular numbers*** as numbers which can represented by triangular-shaped dot patterns.

Ask, "Does $T(n)$ represent a sequence that is arithmetic, geometric, or neither?" Sample response: $T(n)$ is neither arithmetic nor geometric because there is no common ratio or common difference between consecutive terms.

Historical Moment

Have students read the Historical Moment and respond to the given question. Students will learn more about triangular number sequences and work to think of any other shapes that can be formed by number sequences.

Pairs | 10 min

MP7
HSF.BF.A.2, HSF.IF.A.3

Activity 2 Card Sort: Define the Sequence

Students match different types of sequences with their corresponding function rules to reinforce their understanding of recursive rules.

Activity 2 Card Sort: Define the Sequence

You and your partner will be given a set of cards. Match each sequence with its corresponding recursive rule. Note: You are given only part of the sequence.

Sequence	Recursive rule
Card A	Card C
Card B	Card G
Card E	Card I
Card F	Card L

1. Which cards did not have a match? Explain your thinking.

   Cards D, K, J, and H; Sample response:
   - Card D is not recursive.
   - Card K is not recursive.
   - Card J is not recursive.
   - Card H does not include the function's domain.

2. Which cards represent arithmetic sequences? Explain your thinking.

   Sample response: Cards A and C, because each term decreases by a constant rate of 15. Cards E and H, because each term increases by a constant rate of 4. Although Cards D and J are not recursive, they are still arithmetic because the common difference is 2 and $\frac{1}{2}$, respectively.

3. Which cards represent geometric sequences? Explain your thinking.

   Sample response: Cards B and G, because each term decreases by a factor of $\frac{1}{2}$. Cards F and L, because each term increases by a factor of 2.

32 **Unit 1** Sequences and Series

1 Launch

Conduct the ***Card Sort*** routine. Distribute cards from the Activity 2 PDF to pairs of students. Let students know the first term is missing in the recursive rules, so they will need to pay close attention to the relations.

2 Monitor

Help students get started by having them highlight or color code the common difference or ratio of each sequence with the corresponding part of a recursive rule.

Look for points of confusion:

- **Not recognizing cards that are missing the domain restriction or function notation for the previous term.** Ask students to generate a list of values using the given rule to determine whether it makes sense.

Look for productive strategies:

- Removing the cards that do not have the full or correct recursive rule **(MP7)**.
- Using the structure of the equations in function notation to determine the type of sequence.

3 Connect

Display any necessary cards to help facilitate discussion.

Have pairs of students share their strategies for sorting and why they chose not to include certain cards.

Highlight the general form and requirements for defining a sequence recursively. Rules must include the value of the first term, $f(1)$; the current output, $f(n)$, expressed in terms of the previous output, $f(n-1)$; and the possible values of n.

Differentiated Support

Accessibility: *Guide Processing and Visualization*

Consider providing alternative tables, such as the two shown here, for students to use to keep track of their card sorting.

Arithmetic sequence	Recursive rule

Arithmetic sequence	Recursive rule

Math Language Development

MLR7: Compare and Connect

During the Connect, use this routine as students compare strategies for sorting the cards. To support students' reasoning, provide example prompts such as:

- "Why did you include that card?"
- "Why didn't you include that card?"

English Learners: Display the general form for defining a recursive sequence. As students compare their card sort strategies, add annotations to the general form to highlight the requirements for defining a sequence recursively.

Pairs | 10 min

Activity 3 Looking Back to Look Ahead

MP7, MP8
HSF.BF.A.2, HSF.IF.A.3

Students analyze different types of sequences and practice writing recursive rules with an equation by using function notation **(MP8).**

Amps Featured Activity Using Work From Previous Slides

Name: ______ Date: ______ Period: ______

Activity 3 Looking Back to Look Ahead

Complete the equations to write a recursive rule using function notation.

For example: If Sequence P is 30, 40, 50, 60, 70, . . . ,
then $P(1) = 30$ and $P(n) = P(n-1) + 10$, for $n \geq 2$

1. Sequence A: $11, 8, 5, 2, -1, \ldots$

   $A(1) =$ 11 and $A(n) =$ $A(n-1) - 3$, for $n \geq 2$

2. Sequence B: $40, 20, 10, 5, 2.5, \ldots$

   $B(1) =$ 40 and $B(n) =$ $B(n-1) \cdot \frac{1}{2}$, for $n \geq 2$

Use the first five terms of each sequence to determine whether the sequence shown is *arithmetic*, *geometric*, or *neither*. Then write a recursive rule using function notation.

3. Sequence C: $1, 2, 4, 8, 16, \ldots$
   Geometric; $C(1) = 1$ and $C(n) = 2 \cdot C(n-1)$, $n \geq 2$

4. Sequence D: $1, \frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \frac{1}{81}, \ldots$
   Geometric; $D(1) = 1$ and $D(n) = \frac{1}{3} \cdot D(n-1)$, $n \geq 2$

5. Sequence E: $20, 13, 6, -1, -8, \ldots$
   Arithmetic; $E(1) = 20$ and $E(n) = E(n-1) - 7$, $n \geq 2$

6. Sequence F: $1, 3, 7, 15, 31, \ldots$
   Neither; $F(1) = 1$ and $F(n) = 2 \cdot F(n-1) + 1$, $n \geq 2$

STOP

1 Launch

Display the example and highlight the components of a recursive rule. Encourage students to annotate the worked example before continuing the activity.

2 Monitor

Help students get started by asking them to explain the process of writing a recursive rule in their own words. Encourage them to review Activity 2.

Look for points of confusion:

- **Incorrectly writing a recursive rule.** Have students write in words how to generate the next term from the previous one, and then annotate with symbols.

Look for productive strategies:

- Using the structure of recursive rules to write equations for each sequence in function notation **(MP7)**.
- Writing equivalent rules for each sequence. For example,
  $f(n) = \frac{1}{2} f(n-1)$; $f(n) = (0.5) \cdot f(n-1)$;
  $f(n) = \frac{f(n-1)}{2}$

3 Connect

Display the sequences. Record student responses next to each one.

Have pairs of students share their responses. Select and sequence student pairs to describe their equations and reasoning.

Ask, "What do you recall about linear and exponential functions?"

Highlight the connections between arithmetic sequences and linear functions, and the common difference and slope. Also highlight the connections between geometric sequences and exponential functions, and the common ratio and the growth or decay factor.

Differentiated Support

Accessibility: *Optimize Access to Technology*

Have students use the Amps slides for this activity, in which they build on their work from previous slides to help them write recursive rules.

Extension: *Math Enrichment*

Have students complete the following problem:

List the first 5 terms of the sequence defined recursively as

$G(1) = 2$ *and* $G(n) = 3 \cdot G(n-1) + 1$, $n \geq 2$

Then, explain whether the sequence is arithmetic, geometric, or neither.

Sample response: 2, 7, 22, 67, 202. This sequence is neither arithmetic or geometric because there is not a common difference or a common ratio between terms.

Math Language Development

MLR2: Collect and Display

During the Connect, as you highlight the connections between arithmetic sequences and linear functions, and the common difference and slope, add these terms and examples to the class display. Continue to add to the display as you highlight the connections between geometric sequences and exponential functions, and the common ratio and the base of the exponent.

English Learners: Add annotations to the class display that show how arithmetic and geometric sequences are defined recursively and how the recursive rules connect to the linear and exponential functions.

Whole Class | 5 min

Summary

HSF.BF.A.2, HSF.IF.A.3

Review and synthesize that sequences can be defined recursively using function notation, with a domain that is a subset of the integers.

Summary

In today's lesson . . .

You saw that ***triangular numbers*** are numbers that can be represented by triangular dot patterns and can be written as a sequence. Sequences are functions with the domain of whole numbers $1, 2, 3, \ldots$ The domain represents the term number, and it does not make sense to have a term number that is not a whole number. A ***recursive rule*** of a function uses a repeated, or recurring, process to evaluate the function. You can use recursive rules to describe how to calculate the next term in a sequence if you know the previous term (or terms). If n represents the term number, then $f(n)$ denotes the value of the nth term. The previous term number is represented by $n - 1$ and $f(n - 1)$ denotes the value of this term. When writing recursive rules, it is important to define the first term, $f(1)$, and to define the domain as $n \geq 2$, because you know the first term and want to generate the terms that follow.

In general, a recursive rule for an arithmetic sequence is:

$$f(n) = f(n-1) + d, \text{ for } n \geq 2, \text{ where } d \text{ is the common difference.}$$

A general recursive rule for a geometric sequence is:

$$f(n) = f(n-1) \bullet r, \text{ for } n \geq 2, \text{ where } r \text{ is the common ratio.}$$

In each case, $f(1)$ must be given as a starting value so the next term can be calculated.

Reflect:

34 **Unit 1** Sequences and Series

Synthesize

Display the Anchor Chart PDF, *Recursive Rules*.

Highlight that it is important to identify how consecutive terms are changing when writing a recursive rule for a sequence using function notation. In addition, it is important to define a reasonable domain when working with functions so the problem makes sense.

Formalize vocabulary:

- ***recursive rule***
- ***triangular numbers***

Ask, "Why do the recursive rules include $n \geq 2$? Would you be able to write out a sequence from the rule without this part?" Sample response: The term number must be defined and restricted to integer values because it does not make sense to have a decimal or negative number as a term number. You cannot determine values for the sequence recursively without a starting value or first term.

Reflect

After synthesizing the concepts of the lesson, allow students a few moments for reflection on one of the Essential Questions for this unit. Encourage them to record any notes in the *Reflect* space provided in the Student Edition. To help them engage in meaningful reflection, consider asking:

- "Why are sequences considered functions?"

Math Language Development

MLR2: Collect and Display

As students formalize the new vocabulary for this lesson, ask them to refer to the class display for this unit that you started in Lesson 2. Ask them to review and reflect on the terms and phrases related to the term *triangular numbers* that were added to the display during the lesson.

Independent | 5 min

Exit Ticket

HSF.BF.A.2

Students demonstrate their understanding of a recursive rules by interpreting and writing function rules for different types of sequences.

Printable

Name: Date: Period:

Exit Ticket 1.04

Use the first five terms of Sequence H to define the sequence recursively using function notation. Remember to include the domain.

Sequence H: $2.5, 7.5, 22.5, 67.5, 202.5, \ldots$

$H(1) = 2.5$ and $H(n) = H(n-1) \cdot 3$, for $n \geq 2$

Self-Assess

1 I don't really get it | 2 I'm starting to get it | 3 I got it

a I can write a recursive rule using function notation.
1 2 3

b I understand that a recursive rule must include a domain when written in function notation.
1 2 3

Unit 1 Lesson 4 Sequences Are Functions

Success looks like . . .

» **Goal:** Comprehending that sequences are functions whose domain is a subset of the integers.

✓ Using proper function notation.

» Creating a recursive rule for a sequence using function notation.

✓ Writing a recursive rule to represent the given sequence.

Suggested next steps

If students do not include the first term or the domain restriction in the rule, consider:

- Reviewing characteristics of recursive rules in Activity 3.
- Asking, "Would your rule work if the first term of the sequence is 4?"
- Asking, "Would your rule make sense for $n = 0.3$?"

If students write the function equation incorrectly, consider:

- Reviewing the characteristics of recursive rules in Activity 3.
- Assigning Practice Problem 2.
- Having students generate a table for their function to determine whether it matches the given sequence.

Professional Learning

This professional learning moment is designed to be completed independently or collaboratively with your fellow mathematics educators. Prompts are provided so that you can reflect on this lesson before moving on to the next lesson.

Points to Ponder . . .

- What worked and didn't work today? In what ways have your students become better at looking for and making use of structure? **(MP7)**
- Have you changed any ideas you used to have about the recursive rule of a sequence as a result of today's lesson? What might you change for the next time you teach this lesson?

Practice

Independent

Name: ______ Date: ______ Period: ______

Practice

1. Match each sequence with its corresponding rule in function notation. Note that only part of the sequence (showing the relationship between the current term and the previous term) is given.

a. $6, 12, 18, 24$ — b $f(n) = 7 \cdot f(n-1)$

b. $2, 14, 98, 686$ — c $g(n) = \frac{1}{2} \cdot g(n-1)$

c. $160, 80, 40, 20$ — a $h(n) = h(n-1) + 6$

2. List the first five terms of each sequence. Determine whether each sequence is *arithmetic*, *geometric*, or *neither*.

Type of sequence:

a. $a(1) = 7, a(n) = a(n-1) - 3$, for $n \geq 2$

7, 4, 1, −2, −5 — arithmetic

b. $b(1) = 2, b(n) = 2 \cdot b(n-1) - 1$, for $n \geq 2$

2, 3, 5, 9, 17 — neither

c. $c(1) = 3, c(n) = 10 \cdot c\,(n-1)$, for $n \geq 2$

3, 30, 300, 3000, 30000 — geometirc

d. $d(1) = 1, d(n) = n \cdot d(n-1)$, for $n \geq 2$

1, 2, 6, 24, 120 — neither

3. Which function describes the sequence generated by multiplying the previous term by three and then subtracting two?

A. $f(n) = 3 \cdot f(n) - 2$, for $n \geq 2$

B. $f(n) = 3 \cdot (n-1) - 2$, for $n \geq 2$

C. $f(n) = 3 \cdot (n-2)$, for $n \geq 2$

D. $f(n) = 3 \cdot f(n-1) - 2$, for $n \geq 2$

Practice

Name: ______ Date: ______ Period: ______

4. The first five terms of some sequences are given. Describe how to produce the next term.

a. $50, 25, 0, -25, -50$

Subtract 25 from the previous term to get the next term.

b. $\frac{1}{3}, 1, 3, 9, 27$

Multiply the previous term by 3 to get the next term.

c. $2, 4, 6, 8, 10$

Add 2 to the previous term to get the next term.

d. $5, 7, 9, 11, 13$

Add 2 to the previous term to get the next term.

5. Consider the functions $f(x) = 2x - 7$ and $g(x) = 5^x$. Evaluate the terms shown.

a. $f(3), f(2), f(1), f(0), f(-1)$

$-1, -3, -5, -7, -9$

b. $g(3), g(2), g(1), g(0), g(-1)$

$125, 25, 5, 1, \frac{1}{5}$

6. For each sequence, determine the common difference or the common ratio.

a. $5, 15, 20, 25$

The common difference is 5.

b. $f(1) = 7, f(n) = 2 \cdot f(n-1)$, for $n \geq 2$

The common ratio is 2.

c.

n	$g(n)$
1	2
2	10
3	18
4	26

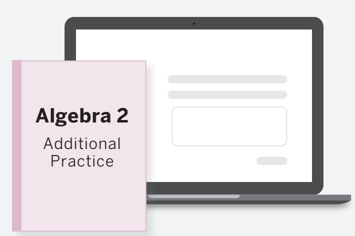
The common difference is 8.

Practice Problem Analysis

Type	Problem	Refer to	Standard(s)	DOK
On-lesson	1	Activity 2	HSF.BF.A.2	1
	2	Activity 3	HSF.BF.A.2	2
	3	Activity 1	HSF.BF.A.2, HSF.IF.A.3	1
Spiral	4	Unit 1 Lesson 3	HSF.BF.A.1.A	2
	5	Algebra 1	HSF.IF.A.2	1
Formative	6	Unit 1 Lesson 5	HSF.LE.A.1	1

Power-up: If students need additional support with the key prerequisite concept or skill this problem addresses, consider assigning the Power-up in the next lesson.

Additional Practice Available



For students who need additional practice in this lesson, assign the **Algebra 2 Additional Practice**.

Representing Sequences

Let's look at different ways to represent a sequence.

Focus

Goals

1. Create a table, graph, or recursive rule of a sequence from given information.
2. **Language Goal:** Determine the information necessary to represent a sequence in different ways and ask questions to elicit it. **(Speaking and Listening, Reading)**

Coherence

Today

Students practice interpreting and writing recursive rules of sequences while also representing sequences in different ways. Students are introduced to the *Info Gap* routine, which allows them to focus on using mathematically precise language as they request necessary information from their partners to analyze and solve problems about sequences **(MP6)**. Throughout the lesson, students are given opportunities to share and explain their strategies for creating different representations and critique the reasoning of others **(MP3)**.

Previously

In Lesson 4, students learned that sequences are functions with a restricted domain and wrote recursive rules for sequences using function notation. Building on their knowledge of different types of functions from Algebra 1, students also made connections between arithmetic sequences and linear functions, and geometric sequences and exponential functions.

Coming Soon

In Lesson 6, students will use technology to reinforce their understanding of recursive rules. In Lesson 7, students will write explicit functions for the nth term of a sequence.

Rigor

- Students develop **procedural fluency** representing sequences with verbal descriptions, tables, graphs, and recursive rules.

Standards

Addressing

HSF.LE.A.2

Construct linear and exponential functions, including **arithmetic and geometric sequences, given a graph, a description of a relationship, or two input-output pairs from a table.**

Also Addressing: **HSF.BF.A.2, HSF.IF.A.3, HSF.IF.C**

Building On	Building Toward
HSF.IF.A.3	HSF.BF.A.2
HSF.IF.B.5	HSF.LE.A.2
HSF.IF.C	

Pacing Guide

Suggested Total Lesson Time **~50 min**

Warm-up	Activity 1	Activity 2	Activity 3	Summary	Exit Ticket
5 min	10 min	10 min	15 min	5 min	5 min
Independent	Pairs	Pairs	Pairs	Whole Class	Independent
MP6	MP3, MP7	MP1	MP1		
HSF.IF.C	HSF.IF.C, HSF.LE.A.2	HSF.BF.A.1.A HSF.IF.C	HSF.BF.A.2, HSF.LE.A.2	HSF.IF.C	HSF.IF.C, HSF.LE.A.2

Amps powered by desmos | **Activity and Presentation Slides**

For a digitally interactive experience of this lesson, log in to Amplify Math at **learning.amplify.com**.

Practice Independent

Materials

- Exit Ticket
- Additional Practice
- Activity 1 PDF, pre-cut cards, one set per pair
- Activity 3 PDF, pre-cut cards, one set per pair
- Activity 3 PDF (answers)
- Instructional Routine PDF, *Info Gap: Instructions*
- Instructional Routine PDF, *Info Gap: Types of Questioning*
- graph paper

Math Language Development

Review words

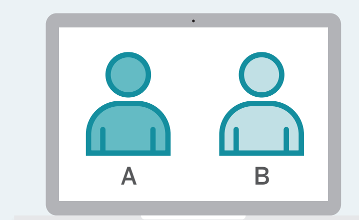
- *arithmetic sequence*
- *geometric sequence*
- *recursive rule*
- *triangular numbers*

Amps | **Featured Activity**

Activity 3

Digital Collaboration

Students are digitally paired to determine and request the information needed to understand how to represent sequences in different ways.



Building Math Identity and Community

Connecting to Mathematical Practices

Students might feel so confident in their own answers that they disregard others and do not listen to their explanations of their choices **(MP3)**. Remind students that maintaining healthy relationships includes listening well to others. Discuss how students can show that they are listening actively. Ask students to identify the benefits of listening to others, even when they already agree on the solution.

Modifications to Pacing

You may want to consider these additional modifications if you are short on time.

- In **Activity 1**, reduce the number of matches by omitting Cards D, H, K and L .
- Omit Problem Card 3 in **Activity 3**.

Independent | 5 min

Warm-up Reading Representations

MP6
HSF.IF.C

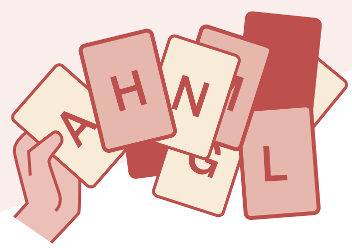
Students identify arithmetic and geometric sequences represented in different ways and make connections to linear and exponential functions.

Name: Date: Period:

Unit 1 | Lesson 5

Representing Sequences

Let's look at different ways to represent a sequence.



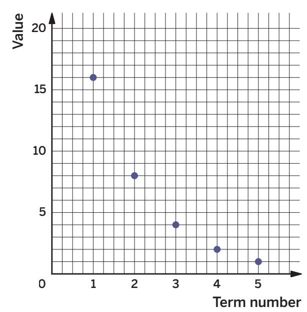
Warm-up Reading Representations

For each sequence, determine either the common ratio or the common difference. Be prepared to explain your thinking.

1. $5, 15, 25, 35, 45$ The common difference is $+10$.

2. Starting at 10, each new term is $\frac{5}{2}$ less than the previous term. The common difference is $-\frac{5}{2}$.

3. The common ratio is 0.5.



4. $g(1) = -5, g(n) = 2 \cdot g(n-1)$, for $n \geq 2$ The common ratio is 2.

5. The common difference is 0.1.

n	1	2	3
$f(n)$	0	0.1	0.2

Log in to Amplify Math to complete this lesson online.

1 Launch

Conduct the ***Think-Pair-Share*** routine.

2 Monitor

Help students get started by reviewing the definitions for *common ratio* and *common difference*.

Look for points of confusion:

- **Misunderstanding how to determine the common ratio or common difference from a graph.** Ask students what the points in a graph of a function represent and prompt them to write out the values of the sequence.

Look for productive strategies:

- Describing the patterns as linear or exponential.

3 Connect

Display Problems 1–5.

Have students share their responses and record them next to each sequence. Select and sequence students who use precise mathematical language to describe the patterns as linear or exponential **(MP6)**.

Highlight that arithmetic sequences have a common difference — or slope — and geometric sequences have a common ratio, in the same way that linear and exponential functions do.

Ask, "In what other ways have you represented functions similar to the examples in the Warm-up?" Functions have been represented by writing equations that describe the output at each input, creating graphs and tables, and giving verbal descriptions.

MLR Math Language Development

MLR2: Collect and Display

During the Connect, collect and display language students use that describes the patterns as linear or exponential. As you add the language to the display, make connections to the previous terms and phrases added to highlight how student language is developing.

Power-up

To power up students' ability to represent a scenario with a function, ask:

A population of 180 bacteria enters a hostile environment, where $\frac{1}{3}$ of the bacteria is destroyed every hour.

a. Write a function representing this scenario. Define your variables.
$f(x) = 180\left(\frac{2}{3}\right)^x$, where x is the number of hours and $f(x)$ is the population of bacteria after x hours.

b. Determine at least 3 ordered pairs which model this scenario.
Sample response: $(0, 180)$, $(1, 120)$, $(2, 80)$

Use: Before the Warm-up
Informed by: Performance on Lesson 4, Practice Problem 6 and the Pre-Unit Readiness Assessment, Problems 3 and 7

Pairs | 10 min

Activity 1 Card Sort: Matching Four Ways

MP3, MP7
HSF.IF.C, HSF.LE.A.2

Students match sequences to their corresponding verbal descriptions, graphs, tables, and recursive rules to reinforce their understanding of the various ways sequences can be represented.

Activity 1 Card Sort: Matching Four Ways

You and your partner will be given a set of cards with multiple representations of different sequences. Sort the cards into four groups so that each group represents the same sequence. Record your sorted cards in the table. Provide an explanation for each sequence modeled that shows why the cards belong together.

Plan Ahead: How did you show others that you were listening to them?

	Cards	Explanation
Group 1	A, I, J, M	This geometric sequence is doubling every term with a common ratio of 2. The first term is 3.
Group 2	B, G, N, P	This arithmetic sequence is increasing every term with a common difference of 5.
Group 3	C, E, F, O	This arithmetic sequence is decreasing every term with a common difference of −5.
Group 4	D, H, L, K	This geometric sequence is doubling every term with a common ratio of 2. The first term is 18.

38 **Unit 1** Sequences and Series

1 Launch

Conduct the ***Card Sort*** routine. Distribute cards from the Activity 1 PDF to pairs of students.

2 Monitor

Help students get started by having them sort cards into arithmetic or geometric categories.

Look for points of confusion:

- **Incorrectly sorting the cards.** Have students sort all the cards with the same representation together (tables, graphs, recursive rules, lists or descriptions). Look for similarities between the different models.

Look for productive strategies:

- Calculating the common difference or common ratio to identify the type of sequence.
- Recognizing linear and exponential patterns in graphs and matching them to their corresponding recursive rules.
- Using structure to distinguish between arithmetic and geometric sequences by the form of their recursive rules **(MP7)**.

3 Connect

Display any necessary cards to facilitate the discussion.

Have pairs of students share their strategies for sorting and why they chose to match certain cards **(MP3)**.

Highlight that it may be more straightforward to discern different information from different representations of sequences.

Ask:

- "What information does each type of representation give you?"
- "Which representation provides the most information? Why?"

Differentiated Support

Accessibility: *Vary Demands to Optimize Challenge*

Consider chunking this task into smaller, manageable parts. For example, instruct students to sort their cards into two groups with a corresponding explanation for each group. Have students verify that their first two sorted groups are accurate before moving on to sort two more groups.

Math Language Development

MLR7: Compare and Connect

During the Connect section, use this routine as students compare strategies for sorting the cards. To support students' reasoning, provide example prompts such as, "Why did you include that card? Why didn't you include that card?"

English Learners: Display the Ask questions. Give students time to consider which representation provides the most information and allow them to look back at how they sorted their cards to make connections to how the different representations made their sorting more or less efficient.

Activity 2 Rectangular Numbers

MP1
HSF.BF.A.1.A, HSF.IF.C

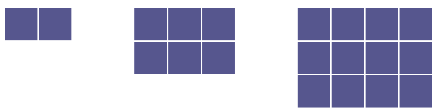
Students create a nonlinear sequence from a mathematical context and practice representing it in various ways.

Name: ________ Date: ________ Period: ________

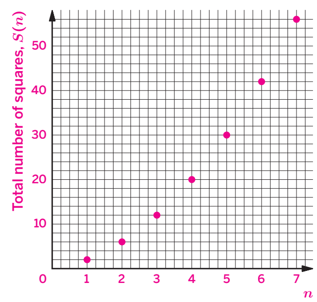
Activity 2 Rectangular Numbers

Consider the following pattern, where the number of unit squares increases with each new figure.

Figure 1 Figure 2 Figure 3



1. List the total number of squares that will appear in Figures 1 through 7, as a sequence with seven terms.
   $2, 6, 12, 20, 30, 42, 56$
2. Write a recursive rule for the total number of squares, $S(n)$, for figure n.
   $S(1) = 2, S(n) = S(n-1) + 2n$, for $n \geq 2$
3. Sketch the graph representing Sequence S.

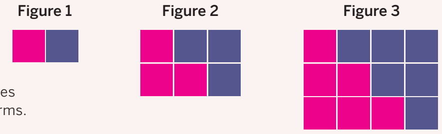


4. Is this sequence *geometric*, *arithmetic*, or *neither*? Explain your thinking.
   Neither. Sample response: The sequence is neither geometric or arithmetic because there is no common difference or common ratio between consecutive figures.

Are you ready for more?

1. Refer to the rectangular pattern from the activity, and shade half of the squares in each figure.

Figure 1 Figure 2 Figure 3



2. List the total number of shaded squares in each figure as a sequence with 7 terms.
   $1, 3, 6, 10, 15, 21, 28$
3. Using what you know about a set of triangular numbers, what is the recursive rule for this sequence, T?
   $T(1) = 1, T(n) = T(n-1) + n$, for $n \geq 2$
4. How does the rule for Sequence T relate to the recursive rule for Sequence S? $T(n) = S(n) \div 2$

1 Launch

Display Figures 1–3. Conduct the ***Notice and Wonder*** routine as students observe and analyze the pattern of figures.

2 Monitor

Help students get started by asking them to determine the additional number of squares added to each figure from the preceding figure.

Look for points of confusion:

- **Assuming the graph in Problem 3 is exponential.** Remind students that one representation of a sequence does not always show the full picture. Have students try to determine a common ratio.

Look for productive strategies:

- Creating additional representations of the sequence.
- Using multiple representations to determine that the sequence is neither arithmetic nor geometric **(MP1)**.

3 Connect

Display the visual pattern and the graph of the sequence.

Have pairs of students share their strategies for determining the recursive rule. Record their responses on the display. Select and sequence pairs that created additional representations.

Highlight that all sequences — not just arithmetic and geometric sequences — are functions and can be represented in various ways.

Ask, "How did you choose the scale for your axes when creating the graph of the sequence?" Sample response: I identified the independent and dependent variables as the term number, n, and value of each term, $S(n)$. The numbers that make sense for the domain are $1, 2, 3, \ldots$ Each tick mark on the horizontal axis represents 1 unit and each tick mark on the vertical axis represents 10 units.

Differentiated Support

Accessibility: *Guide Processing and Visualization*

Provide students with grid or graph paper to organize their work with different representations.

Accessibility: *Clarify Vocabulary and Symbols*

As students complete Problem 4, direct students' attention to the class display and point to the terms *geometric* and *arithmetic*. Highlight the different examples of arithmetic and geometric sequences and ask students to compare the examples on the class display to the current sequence.

Math Language Development

MLR5: Co-craft Questions

Display only the pattern and ask pairs of students to co-craft mathematical questions about the pattern. Give students time to discuss their questions before revealing Problems 1–4.

English Learners: To support students' metalinguistic awareness, demonstrate how to craft a mathematical question and ask students to compare their questions to that of the modeled question. Give students time to revise or improve their initial questions.

Pairs | 15 min

Activity 3 Info Gap: Ways to Represent a Sequence

MP1
HSF.BF.A.2, HSF.LE.A.2

Students build on their understanding of arithmetic and geometric sequences by determining the information needed to represent sequences in different ways and asking questions to gather it **(MP1).**

Amps Featured Activity Digital Collaboration

Activity 3 Info Gap: Ways to Represent a Sequence

You and your partner will be given either a *problem card* or a *data card*. Do not show or read your card to your partner.

If you are given a *problem card*:	If you are given a *data card*:
1. Silently read your card and think about what information you need to be able to solve the problem.	**1.** Silently read your card.
2. Ask your partner for the specific information you need.	**2.** Ask your partner "*What specific information do you need?*" and wait for them to ask for the information.
3. Explain how you are using the information to solve the problem. Continue to ask questions until you have enough information to solve the problem.	**3.** Before sharing the information, ask "*Why do you need that information?*" Listen to your partner's reasoning and ask clarifying questions.
4. Share the *problem card* and solve the problem independently in the space below.	**4.** Read the *problem card* and solve the problem independently in the space below.
5. Read the *data card* and discuss your thinking.	**5.** Share the *data card* and discuss your reasoning.

Pause here so your teacher can review your work. You will be given a new set of cards to repeat the activity, trading roles with your partner.

STOP

40 **Unit 1** Sequences and Series

1 Launch

Conduct the ***Info Gap*** routine and display the Instructional Routine PDF, *Info Gap: Instructions.* Consider demonstrating the routine if students are unfamiliar with it. Provide a Problem Card and Data Card to each pair of students from the Activity 3 PDF along with graph paper. When student pairs complete the first set, have them switch roles and provide another Problem Card and Data Card.

2 Monitor

Help students get started by modeling the questioning process.

Look for points of confusion:

- **Having difficulty discerning which information to use in Data Card 1.** Have students list the different strategies discussed in Activity 1.

Look for productive strategies:

- Generating a list of numbers from a recursive rule.
- Graphing the given points and analyzing the graph.
- Organizing the information they receive to determine whether they have enough information to solve the problem.

3 Connect

Display the cards and solutions to the class.

Have pairs of students share their strategies or any challenges they experienced.

Highlight connections between different representations. For example, a recursive rule might include $f(1) = 16$, which corresponds to the first entry in a table and the point $(1, 16)$ on a graph.

Ask:

- "Which was more difficult: determining what questions to ask, or responding to the questions?"
- "What changes did you make to your questioning and strategies throughout the activity?"
- "What representation of a sequence do you think provides the most information? Why?"

Differentiated Support

Accessibility: *Optimize Access to Technology*

Have students use the Amps slides for this activity, in which they are digitally paired to determine and request the information needed to understand how to represent sequences in different ways.

Extension: *Math Enrichment*

Provide students with a Venn diagram graphic organizer and ask students to complete the Venn diagram by comparing recursive rules and graphs.

Math Language Development

MLR4: Info Gap

This activity uses the ***Info Gap*** to give students a purpose for discussing information necessary to generate terms of a sequence given a function that defines it, or write a recursive function that defines a sequence.

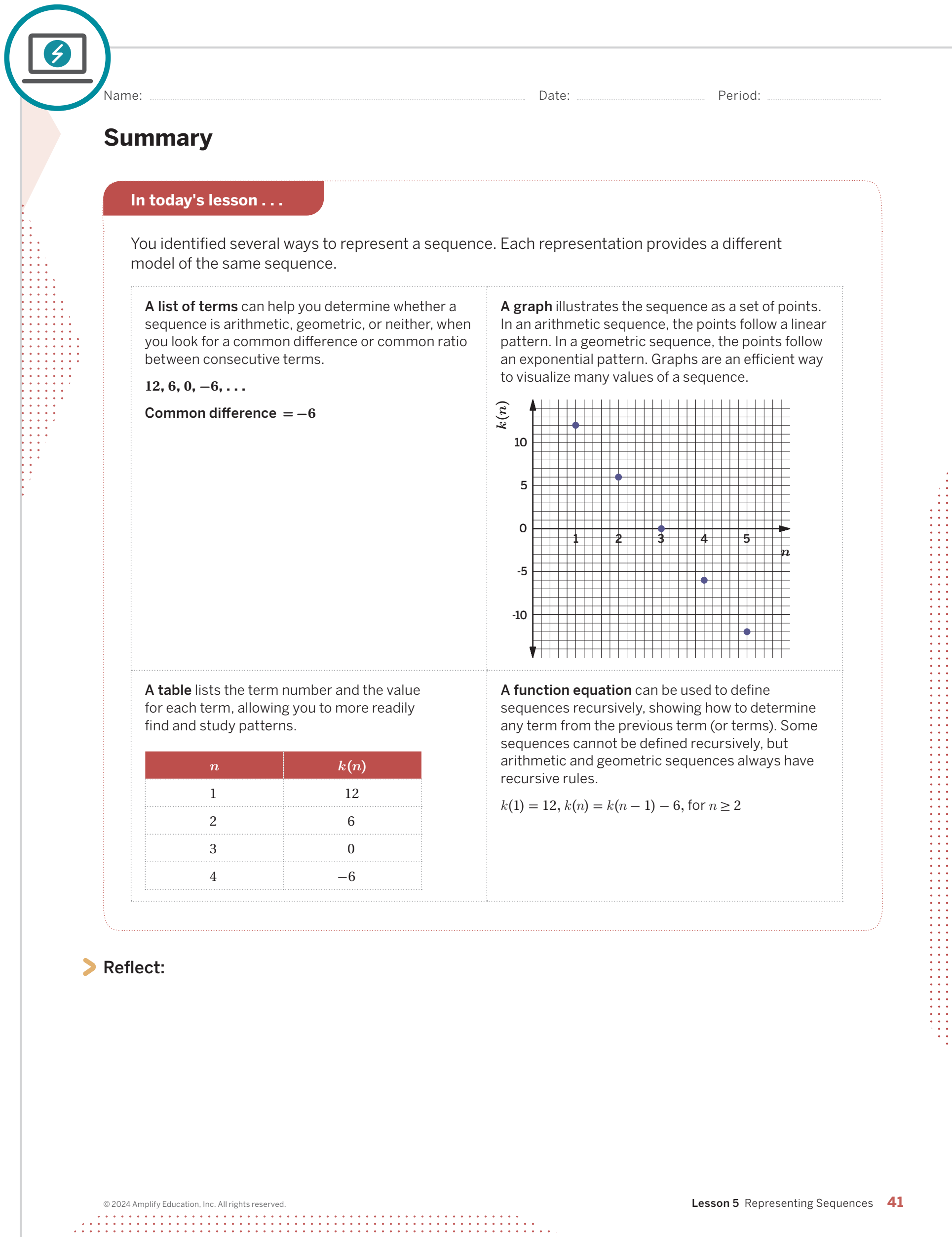
English Learners: Display prompts for students who benefit from a starting point, such as:

- "Can you tell me . . . (specific piece of information)?"
- "Why do you need to know . . . (that piece of information)?"

Summary

HSF.IF.C

Review and synthesize that all sequences are functions and the various ways to represent them.



Name: Date: Period:

Summary

In today's lesson . . .

You identified several ways to represent a sequence. Each representation provides a different model of the same sequence.

A list of terms can help you determine whether a sequence is arithmetic, geometric, or neither, when you look for a common difference or common ratio between consecutive terms.

$12, 6, 0, -6, \ldots$

Common difference $= -6$

A graph illustrates the sequence as a set of points. In an arithmetic sequence, the points follow a linear pattern. In a geometric sequence, the points follow an exponential pattern. Graphs are an efficient way to visualize many values of a sequence.

A table lists the term number and the value for each term, allowing you to more readily find and study patterns.

n	$k(n)$
1	12
2	6
3	0
4	-6

A function equation can be used to define sequences recursively, showing how to determine any term from the previous term (or terms). Some sequences cannot be defined recursively, but arithmetic and geometric sequences always have recursive rules.

$k(1) = 12, k(n) = k(n-1) - 6$, for $n \geq 2$

Reflect:

Synthesize

Display Sequence J from Activity 3 $(2, 8, 32, 128, \ldots)$. Ask students to spend a few minutes representing the sequence in as many ways as they can.

Have students share their reasons for choosing their representations with a partner. Select and sequence students with different representations to share with the whole class.

Highlight that sequences are functions and can be represented in various ways, including a list, table, graph, or recursive rule.

Reflect

After synthesizing the concepts of the lesson, allow students a few moments for reflection. Encourage them to record any notes in the *Reflect* space provided in the Student Edition. To help them engage in meaningful reflection, consider asking:

- "Which representation did you prefer — a table, graph, or equation — when determining the values of a sequence?"

Independent | 5 min

Exit Ticket

HSF.IF.C, HSF.LE.A.2

Students demonstrate their understanding of the various ways to represent a sequence by writing a recursive rule and sketching a graph of a sequence.

Printable

Name: Date: Period:

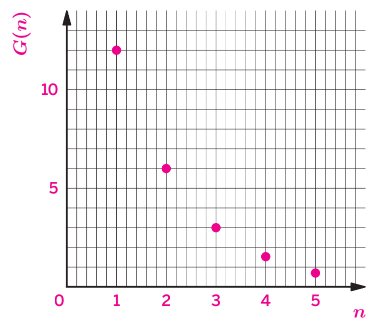
Exit Ticket 1.05

The first two terms of geometric Sequence G are 12 and 6.

1. Write a recursive rule for Sequence G and include the domain.

   $g(1) = 12$, $g(n) = 0.5 \cdot g(n-1)$, for $n \geq 2$

2. Sketch the graph representing the first five terms of Sequence G.



Self-Assess

? 1 I don't really get it | 2 I'm starting to get it | 3 I got it

a I can represent a sequence in different ways, such as by writing a rule and by sketching a graph.
1 2 3

Success looks like . . .

» Creating a table, graph, or recursive rule of a sequence from given information.

✓ Creating a recursive rule in Problem 1 and a graph in Problem 2.

» **Language Goal:** Determining the information necessary to represent a sequence in different ways and asking questions to elicit it. **(Speaking and Listening, Reading)**

Suggested next steps

If students write the recursive rule incorrectly in Problem 1, consider:

- Reviewing recursive rules.
- Assigning Practice Problem 3a.
- Asking, "What is the general format of a recursive rule? How do you produce the next term from the previous term?"

If students graph the sequence incorrectly Problem 2, consider:

- Assigning Practice Problem 1.
- Asking, "What information is necessary to graph a sequence? In what other way could you represent the sequence to obtain that information?"

Professional Learning

This professional learning moment is designed to be completed independently or collaboratively with your fellow mathematics educators. Prompts are provided so that you can reflect on this lesson before moving on to the next lesson.

Points to Ponder . . .

- What worked and didn't work today? How did the ***Info Gap*** routine support students in learning how to represent sequences in different ways?
- During the discussion in Activity 2, how did you encourage each student to share their understandings? What might you change for the next time you teach this lesson?

Math Language Development

Language Goal: Creating a recursive rule in Problem 1 and a graph in Problem 2.

Reflect on students' language development in this lesson towards this goal.

- How have the language routines from Lesson 1 to Lesson 5 helped students develop the language needed to create the recursive rule and graph in this lesson?
- Are there particular routines that have been more helpful? Why or why not?

Practice

Independent

Practice

Name: Date: Period:

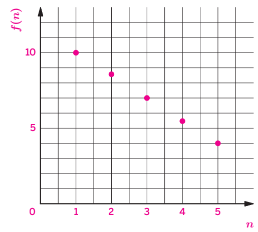
1. Here is the recursive rule for a sequence:

   $f(1) = 10,\ f(n) = f(n-1) - 1.5$, for $n \geq 2$

   a. Is the sequence *arithmetic*, *geometric*, or *neither*?
   Arithmetic

   b. Determine the first five terms of the sequence.
   10, 8.5, 7, 5.5, 4

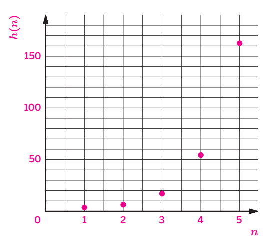
   c. Graph the value of the term, $f(n)$, as a function of the term number, n, for the first five terms of the sequence.



2. The first two terms of a geometric sequence, h, are 2 and 6.

   a. Write a recursive rule for this sequence and include the domain.
   $h(1) = 2,\ h(n) = 3 \cdot h(n-1)$, **for** $n \geq 2$

   b. Graph the first five terms of this sequence.



Name: Date: Period:

Practice

3. The first two terms of a geometric sequence, h, are 80 and 40.

   a. Write a recursive rule for this sequence and include the domain.
   $h(1) = 80,\ h(n) = h(n-1) \cdot \frac{1}{2}$, **for** $n \geq 2$

   b. Use your definition to complete the table for the first six terms of h.

n	1	2	3	4	5	6
$h(n)$	80	40	**20**	**10**	**5**	**2.5**

   c. Explain how to use your recursive rule to determine $h(100)$ without making any calculations.
   Sample response: Starting with 80, divide by 2 a total of 99 times.

4. Match each sequence with its corresponding recursive rule. **Note:** You are given only part of the sequence, the relationship between the current and previous terms.

   a. $3, 15, 75, 375$ — **b** $q(1) = 3,\ q(n) = \frac{1}{3} \cdot q(n-1)$, for $n \geq 2$

   b. $18, 6, 2, \frac{2}{3}$ — **d** $r(1) = 18,\ r(n) = r(n-1) - 4$, for $n \geq 2$

   c. $1, 2, 4, 7$ — **a** $r(1) = 1,\ s(n) = 5 \cdot s(n-1)$, for $n \geq 2$

   d. $17, 13, 9, 5$ — **c** $t(1) = 17,\ t(n) = t(n-1) + (n+1)$, for $n \geq 2$

5. A sequence has $f(1) = 120$ and $f(2) = 60$.

   a. If the sequence is arithmetic, determine the next two terms. Then write a recursive rule.
   $f(3) = 0,\ f(4) = -60,\ f(1) = 120,\ f(n) = f(n-1) - 60$, **for** $n \geq 2$.

   b. If the sequence is geometric, determine the next two terms. Then write a recursive rule.
   $f(3) = 30,\ f(4) = 15,\ f(1) = 120,\ f(n) = f(n-1) \cdot \frac{1}{2}$, **for** $n \geq 2$.

6. Select *all* the expressions that are equivalent to $3 \cdot 3 \cdot 3 \cdot 3$.

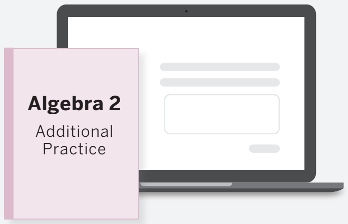
   A. 3^4 B. $3 \cdot 4$ C. $(3^2)^2$ D. $3^2 + 3^2$ E. 4^3 F. $3^2 \cdot 3^2$

Practice Problem Analysis

Type	Problem	Refer to	Standard(s)	DOK
On-lesson	1	Activity 2	HSF.BF.A.1.A, HSF.IF.C	2
	2	Activity 1	HSF.LE.A.2	2
	3	Activity 3	HSF.BF.A.2, HSF.LE.A.2	2
Spiral	4	Unit 1 Lesson 4	HSF.BF.A.2	1
	5	Unit 1 Lesson 4	HSF.BF.A.2, HSF.LE.A.2	2
Formative	6	Unit 1 Lesson 7	8.EE.A.1	1

Power-up: If students need additional support with the key prerequisite concept or skill this problem addresses, consider assigning the Power-up in the next lesson.

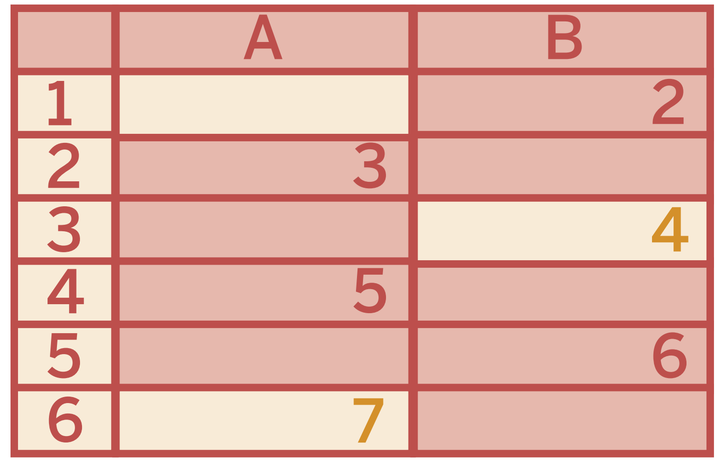
Additional Practice Available



For students who need additional practice in this lesson, assign the **Algebra 2 Additional Practice**.

Using Technology to Work With Sequences

Let's use a spreadsheet to generate the terms of a sequence



Focus

Goal

1. Use a spreadsheet to continue a sequence by applying a recursive rule.

Rigor

- Students develop **procedural fluency** defining sequences recursively with the aid of technology.

Coherence

Today

Students use spreadsheets when working with sequences. In a spreadsheet, students refer to a cell's address, enter a formula that uses the contents of the cell, and generate terms of a sequence.

Previously

In Lesson 5, students defined sequences as *arithmetic*, *geometric*, or *neither*, and represented them in various ways.

Coming Soon

In Lesson 7, students will write explicit definitions for the nth term of a sequence.

Standards

Addressing

HSF.BF.A.1.A
Determine an explicit expression, a recursive process, or steps for calculation from a context.

Also Addressing: **HSF.LE.A.2**

Building On	Building Toward
HSF.LE.A.1	HSF.IF.C
HSF.REI.D.10	

Pacing Guide

Suggested Total Lesson Time ~50 min

Warm-up	Activity 1	Activity 2	Summary	Exit Ticket
5 min	15 min	20 min	5 min	5 min
Independent	Pairs	Pairs	Whole Class	Independent
		MP8		
HSF.BF.A.1.A	HSF.BF.A.1.A, HSF.LE.A.2	HSF.BF.A.1.A, HSF.LE.A.2	HSF.BF.A.1.A, HSF.LE.A.2	HSF.BF.A.1.A

Amps powered by desmos | **Activity and Presentation Slides**

For a digitally interactive experience of this lesson, log in to Amplify Math at **learning.amplify.com**.

Practice Independent

Materials

- Exit Ticket
- Additional Practice
- spreadsheet technology

Math Language Development

Review words

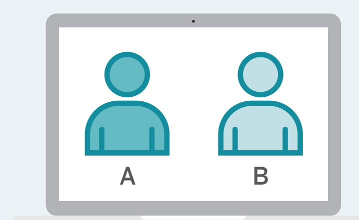
- *arithmetic sequence*
- *geometric sequence*

Amps | **Featured Activity**

Activity 2

Digital Collaboration

Students create their own number pattern using spreadsheet technology and try to guess a classmate's formula.



Amps POWERED BY desmos

Building Math Identity and Community

Connecting to Mathematical Practices

Whenever technology is involved in a lesson, students might be tempted to be distracted and get off task. While the spreadsheet is an amazing tool to help analyze repeated reasoning, it is only as good as the person who is using it **(MP8)**. Before students begin, ask them to identify possible problems that can occur due to bad decisions and what the consequences of those would be. Then have them identify what responsible decision making would look like and how they will employ those techniques.

Modifications to Pacing

You may want to consider this additional modification if you are short on time.

- In **Activity 2**, omit Problems 2, 3, and 5, using spreadsheet technology to analyze geometric sequences.

Independent | 5 min

Warm-up What's the Address?

HSF.BF.A.1.A

Students familiarize themselves with the components of a spreadsheet to prepare for the rest of the lesson.

Unit 1 | Lesson 6

Using Technology to Work With Sequences

Let's a spreadsheet to generate the terms of a sequence.

	A	B
1		2
2	3	
3		4
4	5	
5		6
6	7	

Warm-up What's the Address?

Study the portion of a spreadsheet shown.

100% | $ % .0 .00 123 Default (Ari

fx

	A	B	C	D
1		2		
2	3			
3		4		
4	5			
5		6		
6	7			
7		8		
8	9			
9		10		
10	11			
11				

The space where the rows and columns of a spreadsheet meet are called *cells*. Cells are identified by the column letter and row number of the intersection. This naming convention is an *address*.

1. What are the addresses of the shaded cells?
   A1, A6, B3, B9

2. Which cells have numerical values greater than or equal to 5?
   A4, B5, A6, B7, A8, B9, A10

3. Which cells have numerical values that have a sum of 8?
   Sample responses:
   B1 + B5, A2 + A4, B3 + B3

Log in to Amplify Math to complete this lesson online.

44 Unit 1 Sequences and Series

1 Launch

Instruct students to open spreadsheet technology. Demonstrate how each cell's address is determined (e.g., cell B3 is where the second column and third row meet).

2 Monitor

Help students get started by modeling how to enter various numbers and expressions.

Look for points of confusion:

- **Identifying cell names in each problem.** Annotate the spreadsheet with arrows to show a cell name is built from the intersection of a column and a row.

Look for productive strategies:

- Annotating or creating a table to determine the values that have a sum of 8.

3 Connect

Display the sample spreadsheet.

Have students share their solutions and strategies for Problems 1–3. Record and color code their answers on the spreadsheet.

Highlight the process for reading, writing, and evaluating the cells in a spreadsheet.

Ask, "How have you used spreadsheet technology before?"

Math Language Development

Accessibility: *Guide Processing and Visualization, Vary Demands to Optimize Challenge*

Display a blank spreadsheet on the board for all students to see. Demonstrate how to type in numbers and expressions, as well as how to navigate from cell to cell. Additionally, model for students how to read a cell. For example, click into the cell in the upper left hand corner and highlight for students that this cell is referred to as A1.

Power-up

To power up students' ability to write equivalent expressions using exponents, ask:

Select *all* expressions that are equivalent to $5 \cdot 5 \cdot 5 \cdot x \cdot x \cdot x$.

A. $5^3 \cdot x^3$

B. $3(5x)$

C. $(5x)^3$

D. $\frac{1}{(5x)^{-3}}$

E. $\frac{5}{x^{-3}}$

Use: Before the Warm-up

Informed by: Performance on Lesson 5, Practice Problem 6

Pairs | 15 min

Activity 1 Where Does It Live?

HSF.BF.A.1.A, HSF.LE.A.2

Students interpret and evaluate expressions using cells in a spreadsheet to prepare them for studying the structure of sequences using technology.

Name: ______ Date: ______ Period: ______

Activity 1 Where Does It Live?

Open a blank spreadsheet.

Type the following in each cell:

	A	B
1	2	=A1+A2
2	3	=A3*A4
3	–10	=B1+333
4	0.2	=abs(B2)

1. Look at the values that appear in cells **B1**, **B2**, **B3**, and **B4** after you press *return* or *enter*. How were these values calculated?

	A	B
1	2	5
2	3	–2
3	–10	338
4	0.2	2

Sample response:

- **B1 shows the sum of the numbers in cells A1 and A2.**
- **B2 shows the product of the numbers in cells A3 and A4.**
- **B3 shows the sum of the number in B1 and 333.**
- **B4 shows the absolute value of the number in B2.**

2. Experiment with typing different values into cells **A1**, **A2**, **A3**, and **A4**. Explain what happens.
Sample response: The values in cells B1 through B4 change when the values in cells A1 through A4 change.

3. Experiment with typing new formulas in different cells.

a How can you raise a number to a power?
Sample response: Use the ^ symbol. For example, to compute 2^3 in a cell, type = 2^3.

b What happens if you forget to start a formula with the = symbol?
Sample response: The formula is displayed, but the computation is not completed.

1 Launch

Have students open a blank spreadsheet, providing think-time to read and discuss Problems 1 and 2 with their partners, before releasing them to complete Problem 3 independently.

2 Monitor

Help students get started by modeling how to enter various numbers and expressions into a spreadsheet.

Look for points of confusion:

- **Incorrectly entering the formula.** Remind students they must enter the equal sign (=) first to have the spreadsheet generate a computation.

Look for productive strategies:

- Experimenting with various types of numbers such as integers, positive rational numbers, and negative rational numbers.
- Using various formulas and operations.

3 Connect

Display the spreadsheet.

Have pairs of students share the results of their experiments and model interesting patterns or calculations for the class to see.

Highlight that spreadsheet technology can be useful in calculating expressions and formulas.

Ask, "How might spreadsheets be useful when working with sequences?"

Differentiated Support

Accessibility: *Activate Background Knowledge*

Ask students if they have any experience with budgeting or know of someone who keeps track of their personal finances using a budgeting tool or application. Let students know that many budgeting applications use various forms of spreadsheet technology.

Extension: *Math Enrichment*

Have students complete the following problem:

Create a spreadsheet with at least 4 cells in Column A and Column B. Add values to Column A and formulas to Column B. After creating your spreadsheet, give it to a partner and have them try to determine the formulas you used.

Sample response:

	A	B
1	3	15 = A1 × A2
2	5	2 = A2 – A1
3	6	43 = (A3 × A4) – A2
4	8	$25 = \text{A2}^2$

Math Language Development

MLR8: Discussion Supports

After students complete Problem 2, ask groups to discuss, "What is the relationship between the values in Column A and Column B?" Provide students time to rehearse and formulate a response with a partner before sharing with their group.

English Learners: Encourage students to refer to and use the terms and phrases from the class display to support their discussion.

Pairs | 20 min

Activity 2 Fill Down

MP8
HSF.BF.A.1.A, HSF.LE.A.2

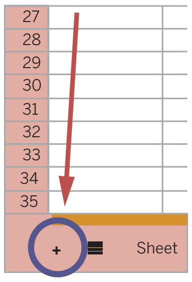
Students generate terms of arithmetic and geometric sequences, while calculating the common difference or common ratio, to understand how spreadsheet technology can help analyze sequences.

Amps Featured Activity | Digital Collaboration

Activity 2 Fill Down

Add a blank sheet by clicking the large + in the bottom left-hand corner.

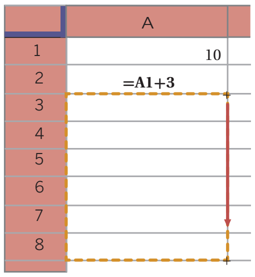
Label the new sheet "Activity 2."



1. Type the value and formula in Column A and *fill down* by selecting the bottom right corner of cell **A2** and dragging down to the bottom of cell **A8**. Explain what happens.
   Each cell shows a value of three more than the cell above it. Column A displays the values 10, 13, 16, 19, . . .

   a. What value could you change in Column A to show the sequence 12, 15, 18, . . .?
   Change the value in cell A1 to 12.

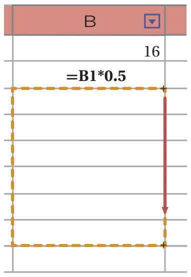
   b. What value could you change in Column A to show the sequence 12, 11, 10, . . .?
   Change cell A2 to =A1–1. Then update using *fill down*.



2. Type the value and formula in Column B and *fill down* to cell **B8**. Explain what happens.
   Each cell shows the product of 0.5 and the cell above it. Column B displays the values 16, 8, 4, 2, . . .

   a. What value could you change in Column B to show the sequence 10, 5, 2.5, . . .?
   Change the value in cell B1 to 10.

   b. What value could you change in Column B to show the sequence 10, 30, 90, . . .?
   Change cell B2 to =B1*3. Then update using *fill down*.



46 Unit 1 Sequences and Series

1 Launch

Demonstrate how to add and label a new spreadsheet. Have students discuss each question prompt with their partner before working independently. Partners should then compare and support their solutions.

2 Monitor

Help students get started by modeling an example similar to Problem 1.

Look for points of confusion:

- **Incorrectly filling in the cells.** Model how to "click and drag down" to copy the formula from the previous cell.
- **Incorrectly calculating common difference or common ratio in the spreadsheet in Problems 3 and 4.** Note whether a cell in the expression is empty, and ask students to write out the formula before typing it in the spreadsheet.

Look for productive strategies:

- Looking for and using more efficient ways to determine each type of sequence.

Activity 2 continued ›

Differentiated Support

Accessibility: *Optimize Access to Technology*

Have students use the Amps slides for this activity, in which they create their own number pattern using spreadsheet technology and try to guess a classmate's formula.

Accessibility: *Vary Demands to Optimize Challenge*

If students need more processing time, have them focus on completing Problems 1–4 and only work on Problems 5–6 if they have time available.

Math Language Development

MLR2: Collect and Display

During the Connect, collect and display language students use to describe how they used the spreadsheet to determine the type of sequence , common ratio and common difference.

English Learners: Clarify the meaning of the phrase "fill down" and explain how this feature connects to arithmetic and geometric sequences for Problems 1 and 2, respectively.

Pairs | 20 min

Activity 2 Fill Down (continued)

MP8
HSF.BF.A.1.A, HSF.LE.A.2

Students generate terms of arithmetic and geometric sequences, while calculating the common difference or common ratio, to understand how spreadsheet technology can help analyze sequences.

Name: ______ Date: ______ Period: ______

Activity 2 Fill Down (continued)

3. Type the values and formulas shown into Columns C and D, and then *fill down* Column D to cell **D7**. What is the result and what does it represent?
   0.1, which is the common ratio for the sequence.

C	D
700	
70	=C2/C1
7	
0.7	
0.07	

4. Type the values and formulas shown into Columns E and F, and then *fill down* Column F to cell **F7**. What is the result and what does it represent?
   3.5, which is the common difference for the sequence.

E	F
7	
10.5	=E2−E1
14	
17.5	

5. Use the spreadsheet to help determine the following:
   a. Is the sequence 8, 12, 18, 27, 40.5 *arithmetic* or *geometric*?
      Geometric
   b. Determine the common difference or common ratio.
      The common ratio is 1.5.

6. Use the spreadsheet to help determine the following:
   a. Is the sequence 50, 42.1, 34.2, 26.3 *arithmetic* or *geometric*?
      Arithmetic
   b. Determine the common difference or common ratio.
      The common difference is −7.9.

Are you ready for more?

Open a new blank sheet. Enter 1 in both cells B1 and B2.

What could you type in cell B3, and then *fill down* to the 10th row, that produces the first 10 terms of the Fibonacci sequence whose first term and second terms are 1?
=B1+B2

STOP

3 Connect

Display examples of student work.

Have students share how they used the spreadsheet to identify each type of sequence and determine the common difference and common ratio.

Highlight that spreadsheet technology is useful when analyzing sequences because it can quickly generate terms. Examining each cell where calculations are made shows that applying the same formula produces different values. Repeatedly using the same operations in a spreadsheet across different cells ensures that the pattern of a sequence extends accurately **(MP8)**.

Whole Class | 5 min

Summary

HSF.BF.A.1.A, HSF.LE.A.2

Review and synthesize how powerful spreadsheet technology can be when generating sequences and determining different types of sequences.

Summary

In today's lesson . . .

You saw that a spreadsheet is a useful tool for creating and analyzing sequences. You saw that when given the first term and the common difference or common ratio, you can type a formula and use the "fill down" function to determine as many more terms as you would like.

You also saw that when a sequence is provided, you can calculate the common difference or common ratio by applying formulas to neighboring cells and looking for successive differences or quotients.

Reflect:

Synthesize

Display a spreadsheet with an arithmetic sequence and common difference calculation.

Have students share their preferred method for:

- Generating the terms of a sequence.
- Calculating the common difference or common ratio.

Highlight that technology is a helpful tool when working with sequences.

Ask, "What are some advantages of using technology to analyze sequences?" Sample response: You can find successive terms quickly by copying the recursive formula and avoid making common mistakes by calculating the math by hand.

Reflect

After synthesizing the concepts of the lesson, allow students a few moments for reflection. Encourage them to record any notes in the *Reflect* space provided in the Student Edition. To help them engage in meaningful reflection, consider asking:

- "When do you think it is appropriate to use technology to analyze sequences?"

Independent | 5 min

Exit Ticket

HSF.BF.A.1.A

Students demonstrate their understanding of using technology to create and analyze sequences.

Printable

Name: ______ Date: ______ Period: ______

Exit Ticket 1.06

1. Describe how to use the spreadsheet shown to generate the first five terms of a geometric sequence that starts with 625 and has a common ratio of $\frac{4}{5}$.

Sample response: I would type 625 into cell A1, then type =A1*(4/5) into cell A2, and then fill down until cell A5.

	A	B	C
1			
2			
3			
4			
5			
6			

2. Describe how to use the spreadsheet shown to determine whether the sequence 111, 103.8, 96.6, 89.4 is arithmetic or geometric.

Sample response: To check whether the sequence is arithmetic, I would type in the sequence in column C and then in cell D1, type =C2−C1 and fill down to determine whether the common difference remains the same. To check whether the sequence is geometric, I would type =C2/C1 in cell E1 and fill down to determine whether the common ratio remains the same.

	C	D	E
1			
2			
3			
4			
5			
6			

Self-Assess

? 1 I don't really get it — 2 I'm starting to get it — 3 I got it

a) I can use a spreadsheet to continue a sequence by applying a recursive rule. 1 2 3

b) I can use a spreadsheet to create many terms of a sequence. 1 2 3

c) I can use a spreadsheet to determine whether a sequence is arithmetic or geometric. 1 2 3

Success looks like . . .

» **Goal:** Using a spreadsheet to continue a sequence by applying a recursive rule.

✓ Explaining the process for generating terms of a sequence in a spreadsheet in Problem 1.

Suggested next steps

If students manually generate each term to solve Problem 1, consider:

- Reviewing "fill down" strategies from Activity 2.
- Asking, "What should you type in cell A2 to generate the terms of a geometric sequence?"

If students manually calculate the common difference to solve Problem 2a, consider:

- Reviewing formulas from Activity 2.
- Asking, "How do you calculate the common difference? What expression can you type into the cell instead?"

Professional Learning

This professional learning moment is designed to be completed independently or collaboratively with your fellow mathematics educators. Prompts are provided so that you can reflect on this lesson before moving on to the next lesson.

Points to Ponder . . .

- What worked and didn't work today? The focus of this lesson was using spreadsheet technology to work with sequences. How did that go?
- What did students find frustrating about working with spreadsheet technology? What helped them work through this frustration? What might you change the next time you teach this lesson?

Practice

Independent

Name: ______ Date: ______ Period: ______

Practice

1. *Technology is required.* Open a blank spreadsheet. In cell **A1**, type 2 and press return.
   a. What should you type into cell **A2** to generate the sequence 2, 4, 8, 16, 32, . . . when you fill down the column?
   =A1*2
   b. What should you type into cell **A2** to generate the sequence 2, 4, 6, 8, 10, . . . when you fill down the column?
   =A1+2

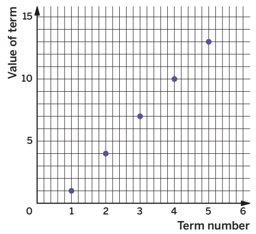
2. *Technology is required.* Open a blank spreadsheet. In cell **A1**, type 400 and press return.
   a. What should you type into cell **A2** to generate the sequence 400, 200, 100, 50, 25, . . . when you fill down the column?
   =A1*0.5
   b. What should you type into cell **A2** to generate the sequence 400, 325, 250, 175, 100, . . . when you fill down the column?
   =A1–75

3. *Technology is required.* Open a blank spreadsheet.
   a. Determine the first five terms of a geometric sequence that starts with –5 and has a common ratio of –1.
   –5, 5, –5, 5, –5
   b. Determine the first five terms of a geometric sequence that starts with –20 and has a common ratio of 0.5.
   –20, –10, –5, –2.5, –1.25
   c. Determine the first five terms of an arithmetic sequence that starts with –20 and has a common difference of 5.
   –20, –15, –10, –5, 0
   d. Determine the first five terms of an arithmetic sequence that starts with 43 and has a common difference of –7.
   43, 36, 29, 22, 15

Practice

Name: ______ Date: ______ Period: ______

4. Refer to the graph of the sequence shown.



   a. Explain how you know this sequence is arithmetic.
   Sample response: The points lie on a line (with a slope of 3). Because the value of each term is 3 more than the previous term, this is an arithmetic sequence.
   b. Explain how you know this sequence is *not* geometric.
   Sample response: This is not a geometric sequence because there is no common ratio between pairs of consecutive terms.

5. The first two terms of a geometric sequence are 6 and 3.
   a. Explain why there is only one geometric sequence with these first two terms.
   Sample response: A geometric sequence is determined by the first term and the common ratio. Here, the first term is 6 and the common ratio is $\frac{1}{2}$. This information produces only one geometric sequence.
   b. What are the next three terms of this geometric sequence?
   $\frac{3}{2}, \frac{3}{4}, \frac{3}{8}$

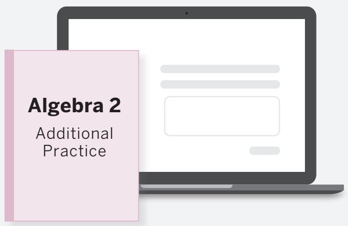
6. Select *all* the expressions that are equivalent to $5 \cdot 5 \cdot 5 \cdot 5 \cdot 5 \cdot 5$.
   A. $5 \cdot 6$
   B. 5^6
   C. $(5^2)^3$
   D. 6^5

Practice Problem Analysis

Type	Problem	Refer to	Standard(s)	DOK
On-lesson	1	Activity 2	HSF.BF.A.1.A, HSF.LE.A.2	2
	2	Activity 2	HSF.BF.A.1.A, HSF.LE.A.2	2
	3	Activity 2	HSF.BF.A.1.A, HSF.LE.A.2	2
Spiral	4	Unit 1 Lesson 5	HSF.IF.C	2
	5	Unit 1 Lesson 2	HSF.LE.A.2	2
Formative	6	Unit 1 Lesson 7	HSF.BF.A.2	1

Power-up: If students need additional support with the key prerequisite concept or skill this problem addresses, consider assigning the Power-up in the next lesson.

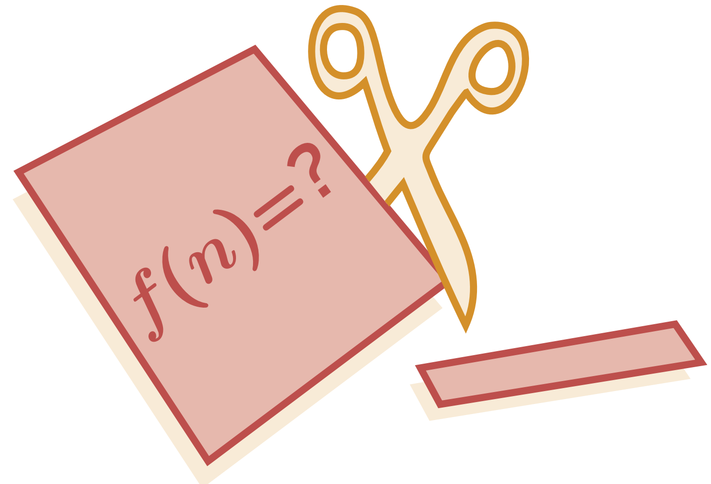
Additional Practice Available



For students who need additional practice in this lesson, assign the **Algebra 2 Additional Practice**.

Finding the nth Term

Let's learn how to define a sequence to determine the nth term.



Focus

Goals

1. Interpret an equation for the nth term of a sequence.
2. Create equations for sequences representing situations.
3. **Language Goal:** Justify why different equations can represent the same sequence. **(Speaking, Listening, Reading and Writing)**

Coherence

Today

Students interpret and write explicit (rather than recursive) rules for the nth term of arithmetic and geometric sequences. A focus of this lesson is using mathematically precise language to explain patterns and to understand how a sequence can be represented by different equations **(MP6)**.

Previously

In Lessons 5 and 6, students wrote recursive rules for sequences and practiced representing sequences in multiple ways.

Coming Soon

In the next several lessons, students will learn how to sum the terms of a sequence and formally define arithmetic and geometric series.

Rigor

- Students develop **procedural fluency** representing sequences with recursive and explicit equations.

Standards

Addressing

HSF.BF.A.2
Write arithmetic and geometric sequences both recursively and with an explicit formula, use them to model situations, and translate between the two forms.

Also Addressing: **HSF.IF.A.3, HSF.IF.B.5, HSF.LE.A.2**

Building On	Building Toward
HSF.IF.C	**HSF.BF.A.2**
HSF.LE.A.1	**HSF.LE.A.2**

Pacing Guide

Suggested Total Lesson Time ~**50 min**

Warm-up	Activity 1	Activity 2	Activity 3 (optional)	Summary	Exit Ticket
5 min	20 min	15 min	15 min	5 min	5 min
Independent	Pairs	Pairs	Pairs	Whole Class	Independent
MP6	MP8	MP6, MP8	MP8		
HSF.BF.A.2, HSF.LE.A.2	HSF.BF.A.2, HSF.IF.A.3, HSF.IF.B.5	HSF.BF.A.2	HSF.BF.A.2, HSF.LE.A.2	HSF.BF.A.2	HSF.BF.A.2

Amps powered by desmos | **Activity and Presentation Slides**

For a digitally interactive experience of this lesson, log in to Amplify Math at **learning.amplify.com**.

Practice Independent

Materials

- Exit Ticket
- Additional Practice
- Anchor Chart PDF, *Explicit Rule for the nth Term*
- scissors (as needed)

Math Language Development

Review words

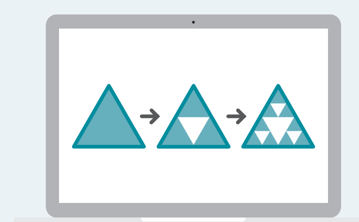
- *exponential function*
- *fractal*
- *linear function*

Amps | **Featured Activity**

Activity 2

Animated Sierpiński Triangle

Students can advance through the stages of this fractal with the click of a button.



Building Math Identity and Community

Connecting to Mathematical Practices

As students share their strategies for determining the different definitions for the number of shaded triangles in the Sierpiński triangle, they might be insensitive to others, whose explanations are different. Students should focus on the mathematical precision of the responses, not the presentation of the content **(MP6)**. While some students might not follow the same social norms, by showing respect and engaging with them, a student might learn and appreciate the differences in social norms among cultures.

Modifications to Pacing

You may want to consider this additional modification if you are short on time.

- Omit optional **Activity 3**.

Independent | 5 min

Warm-up Which One Doesn't Belong?

MP6
HSF.BF.A.2, HSF.LE.A.2

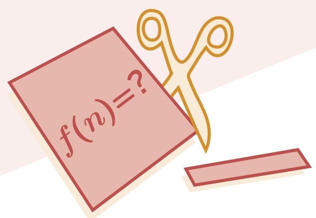
Students compare expressions to practice using mathematical language precisely and prepare them to define sequences in different ways **(MP6).**

Name: Date: Period:

Unit 1 | Lesson 7

Finding the nth Term

Let's learn how to define a sequence to determine the nth term.



Warm-up Which One Doesn't Belong?

Which expression does not belong? Explain your thinking.

Expression A

$5 + 2 + 2 + 2 + 2 + 2 + 2$

Expression B

$5 + 6 \cdot 2$

Expression C

$5 \cdot 2^6$

Expression D

$5 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2$

Sample responses:

- Expression A does not belong because this expression has only addition.
- Expression B does not belong because this expression has both addition and multiplication.
- Expression C does not belong because it is the only expression with an exponent.
- Expression D does not belong because this expression has only multiplication.

Log in to Amplify Math to complete this lesson online.

1 Launch

Conduct the ***Which One Doesn't Belong?*** routine and remind students there are no incorrect answers.

2 Monitor

Help students get started by having them compare the terms of two expressions at a time.

Look for points of confusion:

- Describing multiplication as repeated addition.
- Describing exponents as repeated multiplication.
- Noticing that Expressions A and B are equivalent, as well as Expressions C and D.

3 Connect

Display all four expressions.

Have students share one reason for why an expression does not belong. After each response, ask the class whether they agree or disagree. Select and sequence students who use mathematical language to describe how Expressions A and B are equivalent and Expressions C and D are equivalent.

Highlight that there is no single correct answer, and focus on the reasoning and explanations provided by students. Emphasize that Expressions A and B are equivalent and Expressions C and D are equivalent.

Ask:

- "How are Expressions A and B similar?"
  Sample response: They are both equal to 17.
- "How are Expressions C and D similar?"
  Sample response: They are both equal to 320.

MLR Math Language Development

MLR2: Collect and Display

As students describe which expression doesn't belong, collect and display language that students use to explain their thinking. For example, amplify terms and phrases that connect Expressions A and B, and Expressions C and D.

Power-up

To power up students' ability to write equivalent expressions using exponents, ask:

1. Write an expression as a power of 3, that is equivalent to $3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3$.
   3^6
2. Write an expression using a power of 3, that is equivalent to $2 \cdot 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3$.
   2×3^6

Use: Before the Warm-up

Informed by: Performance on Lesson 6, Practice Problem 6 and the Pre-Unit Readiness Assessment, Problem 4

Pairs | 20 min

Activity 1 Any Way You Slice It

MP8
HSF.BF.A.2, HSF.IF.A.3, HSF.IF.B.5

Students write recursive and explicit rules for an arithmetic sequence, analyze domain in context, and make connections to the equation of a linear function.

Activity 1 Any Way You Slice It

Clare has an 8 in. by 10 in. piece of paper. She cuts off one inch of the width — a strip that is 8 in. by 1 in. Then she cuts off another strip of paper the same size, and repeats this process.

1. Complete the table showing the area of Clare's remaining paper C in square inches, as a result of the previous step (the second column) and in terms of the number of cuts n (third column).

Number of cuts, n	Area in terms of the previous area (in^2)	Area in terms of n (in^2)
1	72	$72 - 8 \cdot$ **$(1 - 1)$**
2	$72 - 8 =$ **64**	$72 - 8 \cdot$ **$(2 - 1)$**
3	**64** $-$ **8** $= 56$	$72 - 8 \cdot$ **$(3 - 1)$**
4	**56** $- 8 =$ **48**	$72 - 8 \cdot$ **$(4 - 1)$**
5	**48** $-$ **8** $=$ **40**	$72 - 8 \cdot$ **$(5 - 1)$**

2. State the area of the paper remaining after each cut as a sequence with five terms and write a recursive rule for $C(n)$.
$72, 64, 56, 48, 40$; $C(1) = 72$, $C(n) = C(n - 1) - 8$, for $2 \leq n \leq 10$

3. Is this sequence arithmetic, geometric, or neither? Explain your thinking.
Arithmetic; Sample response: The sequence is arithmetic because there is a common difference, $d = -8$, between consecutive terms.

4. Write an equation for $C(n)$, the area of the paper after n cuts. This equation should *not* be the same as your recursive rule, and should include $C(1)$ and the common difference. To help with your thinking, review the last column of the table in Problem 1.
Sample response: $C(n) = 72 - 8(n - 1)$

5. What is a reasonable domain for the function in Problem 4? Explain your thinking.
Sample response: The possible values for n are whole numbers from 0 to 10. It does not make sense to make a partial cut and the function is not defined for more than 10 cuts because there would be no paper remaining to make another cut.

52 **Unit 1** Sequences Are Functions

1 Launch

Set an expectation for the amount of time students will have to work in pairs on the activity.

2 Monitor

Help students get started by asking them to sketch the process of cutting each strip of paper, or use scissors to perform each cut, and then determine the remaining area.

Look for points of confusion:

- **Stating the first term or the domain incorrectly.** Ask students to use their recursive rule to generate a sequence, and then see whether it matches the values in the table.

Look for productive strategies:

- Using repeated reasoning to write both recursive and explicit rules for function C **(MP8)**.
- Noticing that the explicit rule for function C is the same as $y = mx + b$, where $m = -8$ and $b = 80$.

Activity 1 continued >

Differentiated Support

Accessibility: *Clarify Vocabulary and Symbols*

Display review vocabulary terms and their definitions, such as *domain* — the set of possible x-values — to help students understand that a reasonable domain for the function is positive whole numbers between 0 and 10.

Accessibility: *Vary Demands to Optimize Challenge*

Consider chunking this task into smaller, manageable parts. For example, after students complete the table in Problem 1, review the solutions as a whole class. Then, check in with students after they complete Problem 4 to ensure pairs of students are in agreement before moving on to the remaining problems.

Math Language Development

MLR7: Compare and Connect

Have pairs of students compare similarities and differences between recursive and explicit rules. Make explicit connections to the different parts of the equation in Problem 5 and how that relates directly to creating the equation in Problem 6 in order to answer Problem 7.

English Learners: As students compare similarities and differences, remind them to use language from the class display to support their use of appropriate math language.

Activity 1 Any Way You Slice It (continued)

MP8
HSF.BF.A.2, HSF.IF.A.3, HSF.IF.B.5

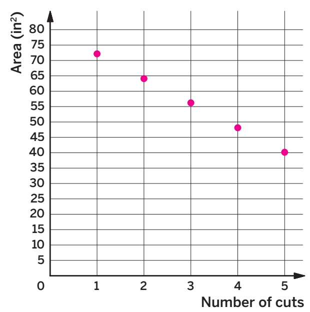
Students write recursive and explicit rules for an arithmetic sequence, analyze domain in context, and make connections to the equation of a linear function.

Name: ______ Date: ______ Period: ______

Activity 1 Any Way You Slice It (continued)

6. Compare the domains from Problems 2 and 5. Are they the same or different? Explain your thinking.
Sample response: They are represented differently, but show the same values of whole numbers from stage 1 to 10. The domain in the explicit form includes 0, which makes sense in the context of having 0 cuts.

7. Graph the number of cuts and the area of paper.



8. Elena wrote the linear function $C(n) = 80 - 8n$, where $1 \leq n \leq 10$. Do you agree with Elena's equation? Explain your thinking.
I agree; Sample response: The two equations are equivalent and represent a linear function with a slope of -8 in² per cut and a y-intercept of 80 in², the area of the original piece of paper.

9. Clare says that after 6 cuts, the area is $80 - 8 \cdot 6$. Explain what each part of the expression represents.
Sample response: Each cut removes 8 in² from the paper, so 6 cuts removes $8 \cdot 6$ in². The original area of the paper is 80 in², so the remaining area can be determined by $80 - 8 \cdot 6$.

10. After 8 cuts, what is the area of paper left? Show your thinking. Did you use the recursive rule or your equation in Problem 4?
16 in²; $80 - 8 \cdot 8 = 80 - 64 = 16$ in²

3 Connect

Display the table from Problem 1 and part of the Anchor Chart PDF, *Explicit Rule for the nth Term*. Do not display the equation for the nth term of a geometric sequence.

Have pairs of students share their recursive and explicit rules for the remaining area of paper. Record their responses next to the table.

Highlight that the recursive rule and the explicit rule for the nth term both represent the remaining area of paper. Emphasize that the rule can be represented differently depending on the starting term number. For instance, the table starts with Term 1 at 72, but this sequence could start with Term 0 at 80, which makes sense in this context. The equation for the nth term is equivalent to the equation of a linear function where the common difference represents the slope and "Term 0" represents the y-intercept.

Ask:

- "If the values for the area of the paper are represented as a sequence, does it matter if the first row is considered as 'Term 0' or as 'Term 1'? Which term name makes the most sense?" Sample response: It does not matter if the first row is labeled "Term 0" or "Term 1" because the values of the sequence stay the same. However, the domain would need to be defined differently. Here, "Term 0" makes more sense because zero cuts have been made at the start of the activity.
- "Do the domains for the recursive and explicit rules have any restrictions? Why or why not?" Sample response: Realistically, the paper cannot be cut into more than 9 strips. Therefore, you can say the domain is all integers from 0 to 9.

Pairs | 15 min

MP6, MP8
HSF.BF.A.2

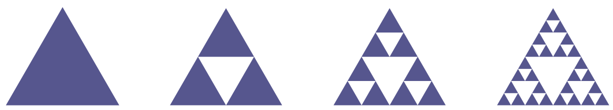
Activity 2 The Sierpiński Triangle

Students write recursive and explicit rules for a geometric sequence and make connections to the equation for an exponential function.

Amps Featured Activity Animated Sierpiński Triangle

Activity 2 The Sierpiński Triangle

The Sierpiński triangle is a fractal, or a pattern that repeats itself as you zoom in or out. This pattern can be found in the natural world, such as on the outer shell of seashells. You can make the Sierpiński triangle by starting with an equilateral triangle whose area is 1 square unit. Divide it into 4 congruent triangles and remove the middle triangle. Then repeat this process with each of the remaining triangles, and so on. The following figure shows the first several steps of this construction as the process is repeated.



Step 0 Step 1 Step 2 Step 3

1. Complete the table that shows the number of shaded triangles, B, as a result of the previous step.

Step, n	Number of shaded triangles in terms of the previous step	Number of shaded triangles in terms of n
0	1	$1 \cdot 3^{\boxed{0}}$
1	$1 \cdot$ 3 = 3	$1 \cdot 3^{\boxed{1}}$
2	$3 \cdot$ 3 = 9	$1 \cdot 3^{\boxed{2}}$
3	9 $\cdot$ 3 = 27	$1 \cdot 3^{\boxed{3}}$

2. Write a recursive rule for Function B.
   $B(0) = 1,\ B(n) = 3 \cdot B(n-1)$, for $n \geq 1$

3. Is this sequence *arithmetic*, *geometric*, or *neither*? Explain your thinking.
   Geometric; Sample response: The sequence is geometric because the number of shaded triangles grows by a factor of 3 each step.

4. Write a rule for Function B, that is *not* recursive, and gives the number of shaded triangles at step n. Be sure to include the domain.
   $B(n) = 1 \cdot 3^n$, for $n \geq 0$

5. How is the equation you wrote in Problem 4 similar to the equation for an exponential function?
   The equation for an exponential function is $f(x) = a \cdot b^x$, where a is the y-intercept and b is the growth or decay factor.

6. Han numbered the first triangle as Step 1 and wrote the sequence as $B(n) = 1 \cdot 3^{n-1}$, where $n \geq 1$. Do you agree with Han? Explain your thinking.
   Sample response: Yes; the function produces the same values but its domain starts at 1 instead of 0.

Reflect: What did you do to show sensitivity to another? How did your expectations for the presentation differ?

54 **Unit 1** Sequences and Series

1 Launch

Display the diagram of how to create a Sierpiński triangle. Activate prior knowledge about how to create a Sierpiński triangle.

2 Monitor

Help students get started by completing the first row of the table.

Look for points of confusion:

- **Stating the first term or the domain incorrectly.** Ask students to use their recursive rule to generate a sequence and determine whether it matches the values in the table **(MP6)**.

Look for productive strategies:

- Using repeated reasoning to write a recursive and explicit rules for function B **(MP8)**.
- Noticing that the explicit rule for function B is equivalent to $y = a \cdot b^x$, where $b = 3$ and $a = 1$.

3 Connect

Display a digital animation of the figures approaching the Sierpiński triangle. Next, display the table from Problem 1 and the Anchor Chart PDF, *Explicit Rule for the nth Term*.

Have students share their strategies for determining the recursive and explicit rules that represent the number of shaded triangles in each figure. Select and sequence students to share their thinking. Record their responses next to the table.

Highlight that the recursive and explicit rules both represent the number of shaded triangles. The equation for the nth term is equivalent to the equation of an exponential function, where the common ratio represents the growth factor and "Term 0" represents the y-intercept.

Ask, "How would the explicit definition change if the pattern started with Step 1 instead of Step 0?"
$B(n) = 3^{n-1}$ for $n \geq 1$

Differentiated Support

Accessibility: *Optimize Access to Technology*

Have students use the Amps slides for this activity, in which they can advance through the stages of the Sierpiński Triangle fractal with the click of a button.

Extension: *Math Enrichment*

Have students complete the table and write a rule for the function:

n	$f(n)$
0	2
1	$2 \cdot 3 = 6$
2	18
3	54

$f(n) = 2 \cdot 3^n$

Math Language Development

MLR1: Stronger and Clearer Each Time

Use this routine to prepare students for the Connect discussion by providing them with multiple opportunities to clarify their explanations through conversation. During Problem 5, provide students time to create a draft response before meeting with 2–3 partners to receive feedback on their response. After receiving feedback, have students use the feedback to revise and improve upon their draft response.

Optional

Pairs | 15 min

Activity 3 Fibonacci Revisited

MP8
HSF.BF.A.2, HSF.LE.A.2

Students revisit the Fibonacci sequence to explore how to define nonlinear sequences, recognizing that it is not always straightforward to write an explicit rule.

Name: ______ Date: ______ Period: ______

Activity 3 Fibonacci Revisited

Recall the Hemachandra (or Fibonacci) sequence from earlier in this unit:
1, 1, 2, 3, 5, 8, . . .

1. What are the next two terms of this sequence? **13, 21**

2. If $F(1) = 1$ and $F(2) = 1$, write a recursive rule to determine the current term, $F(n)$, using the previous two terms. Be sure to include the domain.
$F(n) = F(n-1) + F(n-2)$, for $n \geq 3$

3. Complete the table for the first eight terms of the Fibonacci sequence. Then complete the following problems.

a. Is this sequence *arithmetic*, *geometric*, or *neither*? Explain your thinking.
Neither; Sample response: This sequence is not arithmetic because the common difference changes, and it is not geometric because the common ratio changes. The amount added to produce the next number grows larger as the sequence continues. It is not geometric because the ratio between consecutive terms changes.

b. Can you write a rule to determine the nth term of the Fibonacci sequence? Why or why not?
No; Sample response: Because the sequence is not arithmetic or geometric and has no obvious pattern that relates the term number to each value of the sequence.

n	$F(n)$
1	1
2	1
3	2
4	3
5	5
6	8
7	13
8	21

STOP

1 Launch

Activate prior knowledge by referring to the Hemachandra (or Fibonacci) sequence from the Launch lesson.

2 Monitor

Help students get started by asking them to describe in words how to determine the next term in the sequence.

Look for points of confusion:

- **Misinterpreting how to express the two previous terms, which is $F(n–2)$.** Have students represent the "current term" and the "previous term" in function notation.
- **Stating the domain incorrectly.** Ask students to use their recursive rule to generate a sequence and determine whether it matches the Fibonacci sequence.

Look for productive strategies:

- Using repeated reasoning to write a recursive rule for function F **(MP8)**.

3 Connect

Display the table in Problem 3 with solutions and the recursive rule for function F.

Have pairs of students share their responses to Problem 3, parts a and b.

Highlight that the nth term of the Fibonacci sequence is complicated and was not discovered until the 17th century. Display:

$$F(n) = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right)$$

Ask:

- "Which definition is more convenient for determining $F(5)$? What about $F(100)$?" Sample response: Answers may vary, though many will likely prefer the recursive rule for $F(5)$ because it is simple. The explicit rule is the faster way to calculate $F(100)$.
- "What is the domain of the function that represents the Fibonacci sequence? Does it have any restrictions?" Sample response: The domain is $n \geq 1$. It does not have an upper bound because it goes on forever.

Differentiated Support

Accessibility: *Activate Prior Knowledge*

Connect this lesson to the Launch lesson students completed in Lesson 1 and remind students of the Featured Mathematician, Acharya Hemachandra, and his pattern. Display the Hemachandra sequence before beginning the lesson.

Math Language Development

MLR3: Critique, Correct, Clarify

Before students answer Problem 3a, present an incorrect argument, such as "The sequence is geometric because the common difference changes," and then have students critique the statement, correct it, and clarify how and why they corrected it.

English Learners: Give students time to rehearse and formulate a response with a partner before sharing with the whole class.

Whole Class | 15 min

Summary

HSF.BF.A.2

Review and synthesize how to define sequences explicitly using function notation, how different equations can represent the same sequence, and how to choose an appropriate domain.

Summary

In today's lesson . . .

You defined sequences recursively by looking at the previous term of a sequence to produce the next term. When determining the values of terms further in a sequence, such as the 20th term, you need a more efficient method. Writing a rule for the nth term of a sequence allows you to substitute any term number into a function to determine the value of the term. For an arithmetic sequence, the nth term is calculated by using a linear function, and for a geometric sequence, the nth term is calculated by using an exponential function.

Finding the nth term in an arithmetic sequence	Finding the nth term in a geometric sequence
$a(n) = a(1) + (n - 1)d$ where $a(1)$ is the first term and d is the common difference	$a(n) = a(1) \cdot r^{n-1}$ where $a(1)$ is the first term and r is the common ratio
$7, 9, 11, 13, \ldots$ $f(n) = 7 + (n - 1) \cdot 2$, for $n \geq 2$ $f(1) = 7$ $d = 2$ This can be rewritten as the linear function $f(n) = 5 + 2n$, where the slope is 2 and the y-intercept is $(0, 5)$.	$160, 40, 10, 2.5, \ldots$ $f(n) = 160 \cdot (0.25)^{n-1}$, for $n \geq 2$ $f(1) = 160$ $r = 0.25$ This can be rewritten as the exponential function $f(n) = 640 \cdot (0.25)^n$, where the decay rate is 0.25 and the y-intercept is $(0, 640)$.

The first term of a sequence can be represented by $n = 0$ or $n = 1$, depending on the real-world scenario, and the domain represents any limitations to the context. Some sequences are neither arithmetic nor geometric, and cannot be described with a rule for the nth term, such as the Fibonacci sequence.

Reflect:

56 Unit 1 Sequences and Series

Synthesize

Display the Anchor Chart PDF, *Explicit Rule for the* n*th Term.*

Have students share their thoughts on when to use an explicit rule versus a recursive rule for a sequence and how to define the domain.

Highlight that both recursive and explicit rules are useful. Determining the nth term is convenient when you want to calculate a value with a large term number. A recursive rule is helpful if you already know the previous term in the sequence. Domains are determined by the context of the problem and how the sequence is defined.

Ask, "In what situations does it make sense to start with Term 0, and when is it more useful to start with Term 1?"

Reflect

After synthesizing the concepts of the lesson, allow students a few moments for reflection. Encourage them to record any notes in the *Reflect* space provided in the Student Edition. To help them engage in meaningful reflection, consider asking:

- "What are some things to keep in mind when determining which rule to use to represent a sequence?"

Independent | 5 min

Exit Ticket

HSF.BF.A.2

Students demonstrate their understanding of how to explicitly define a sequence with an appropriate domain and how different rules can represent the same sequence.

Printable

Name: ______ Date: ______ Period: ______

Exit Ticket 1.07

1. A sequence is defined by the rule $f(1) = -2.5$, $f(n) = f(n-1) + 0.5$, for $n \geq 2$.

   Elena and Diego both write a function for determining the nth term of the sequence:

Elena's rule	Diego's rule
$f(n) = -2.5 + 0.5(n-1)$	$f(n) = -3 + 0.5n$

   Are they correct? Explain your thinking.

   Yes; Sample response: Both equations create a sequence where Term 1 is –2.5 and increases by 0.5 with each term. These equations are equivalent to each other.

2. Consider the following geometric sequence: $2, 8, 32, 128, \ldots$

   What is the equation of the function for the nth term?

   Sample response: $g(n) = 2(4)^{n-1}$ for $n \geq 2$.

Self-Assess

? 1 I don't really get it | 2 I'm starting to get it | 3 I got it

a. I can define a sequence using an equation. 1 2 3

b. I can explain why different equations can represent the same sequence. 1 2 3

c. I can represent situations with sequences. 1 2 3

Unit 1 Lesson 7 Finding the nth Term

Success looks like . . .

» **Goal:** Interpreting an equation for the nth term of a sequence.

✓ Recognizing equivalent rules (recursive and explicit) in Problem 1.

» **Goal:** Creating equations for sequences representing situations.

✓ Determining an explicit rule in Problem 2.

» **Language Goal:** Justifying why different equations can represent the same sequence. **(Speaking, Listening, Reading and Writing)**

✓ Explaining thinking in Problem 1.

Suggested next steps

If students cannot complete Problem 1, consider:

- Reviewing Activity 1.
- Assigning Practice Problem 2.
- Having students use different representations to compare the rules such as tables or graphs.

If students cannot write the rule for the nth term in Problem 2, consider:

- Reviewing Activity 2.
- Assigning Practice Problem 3.

Professional Learning

This professional learning moment is designed to be completed independently or collaboratively with your fellow mathematics educators. Prompts are provided so that you can reflect on this lesson before moving on to the next lesson.

Points to Ponder . . .

- What worked and didn't work today? In this lesson, students learned how to write explicit rules for the nth term of arithmetic and geometric sequences. How will that support upcoming lessons on working with series?
- What resources did students use as they worked on writing explicit rules for sequences? Which resources were especially helpful? What might you change for the next time you teach this lesson?

Math Language Development

Language Goal: Justifying why different equations can represent the same sequence

Reflect on students' language development in this lesson towards this goal.

- In what ways did students use their developing math language to justify and explain their thinking?
- What support do they still need in order to be more precise in their justifications?

Sample descriptions:

Emerging	Expanding
The sequences are the same because they both start with the same number and grow by the same amount.	The sequences are the same because they have the same initial value and increase by the same common difference.

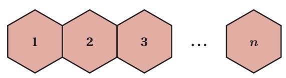
Practice

Independent

Name: ______ Date: ______ Period: ______

Practice

1. A party layout shows hexagonal tables placed together with space for one person on each open side:



   a. Complete the table to represent the number of people $P(n)$, who can sit at n tables.

n	1	2	3	4	5
$P(n)$	**6**	**10**	**14**	**18**	**22**

   b. Describe how the number of people who can sit at the tables changes with each step.

   Sample response: The number of people who can sit at the tables increases by 4 each time a table is added.

   c. Write a recursive rule for P and for the nth term.

   $P(1) = 6, P(n) = P(n-1) + 4,$ **for** $n \geq 2$; $P(n) = 6 + 4(n-1),$ **for** $n \geq 1$

   d. Explain why $P(3.2)$ does not make sense in this scenario.

   Sample response: $P(3.2)$ would represent the number of people who can sit at 3.2 tables, which does not make sense, because there are only whole tables added each step.

2. Here is a recursive rule for a sequence: $f(0) = 19, f(n) = f(n-1) - 6,$ for $n \geq 1$. The function for the nth term of the sequence is $g(n) = 19 - 6 \cdot n,$ for $n \geq 0$.

   a. Explain how you know that functions f and g both represent the same sequence.

   Sample response: Functions f and g both represent the same sequence because they each start at 19 and every term decrease by 6.

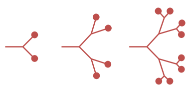
   b. Select function f or g to calculate the 20th term and explain why you chose it.

   Sample response: $g(19) = -95$; I used the function for the nth term because it is the easiest method to determine the 20th term.

Practice

Name: ______ Date: ______ Period: ______

3. Consider the growing pattern shown.



   a. Describe how the number of dots increases from one figure to the next and write a function for the nth term of the sequence $D(n)$, which represents the number of dots in each stage.

   Sample response: From one step to the next, the number of dots doubles.
   $D(n) = 2 \cdot (2)^n$, **for** $n \geq 0$

   b. What is a reasonable domain for function $D(n)$? Explain your thinking.

   $n \geq 0$; Sample response: The domain represents the stage number. If the first image is "Stage 0", then the domain is $n \geq 0$.

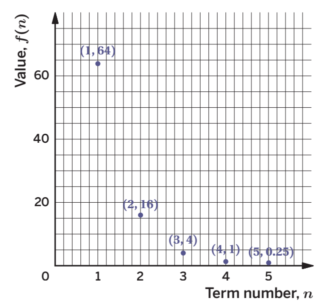
4. Write the first five terms of each sequence. Then determine whether each sequence is *arithmetic*, *geometric*, or *neither*.

   a. $p(1) = 5, p(n) = p(n-1) + 3,$ for $n \geq 2$
   5, 8, 11, 14, 17; arithmetic

   b. $q(1) = 1, q(n) = 3 \cdot q(n-1),$ for $n \geq 2$
   1, 3, 9, 27, 81; geometric

   c. $r(1) = 3, r(n) = -r(n-1) + 1,$ for $n \geq 2$
   3, −2, 3, −2, 3; neither

5. The graph of a sequence is shown.



   a. Is this sequence *arithmetic* or *geometric*? Explain your thinking.

   Geometric. Sample response: There is a common ratio of $\frac{1}{4}$ from one term to the next, so the sequence is geometric.

   b. Write the first five terms of the sequence.

   64, 16, 4, 1, $\frac{1}{4}$

   c. Write a recursive rule for the sequence.

   $f(1) = 64, f(n) = \frac{1}{4} \cdot f(n-1),$ **for** $n > 2$

6. A rabbit population is growing in such a way that each new generation is 2 times larger than the previous generation. Suppose there are 10 rabbits in the first generation. How many total rabbits will there be in 4 generations combined?

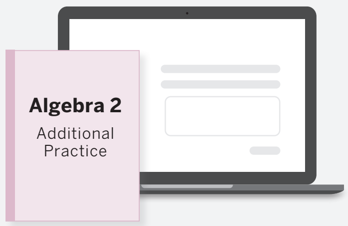
   $10 + 20 + 40 + 80 = 150$ **rabbits**

Practice Problem Analysis

Type	Problem	Refer to	Standard(s)	DOK
On-lesson	1	Activity 1	HSF.BF.A.2, HSF.IF.B.5	2
	2	Activity 1	HSF.BF.A.2	2
	3	Activity 3	HSF.BF.A.2	2
Spiral	4	Unit 1 Lesson 4	HSF.BF.A.2	2
	5	Unit 1 Lesson 5	HSF.LE.A.2	2
Formative	6	Unit 1 Lesson 11	HSF.LE.A.1.B, HSF.LE.A.1.C	2

Power-up: If students need additional support with the key prerequisite concept or skill this problem addresses, consider assigning the Power-up in the next lesson.

Additional Practice Available



For students who need additional practice in this lesson, assign the **Algebra 2 Additional Practice.**

UNIT 2 | **LESSON 2**

Evaluating Polynomials

Let's look at some other situations that polynomials can model.

Focus

Goals

1. Create a polynomial to model a simple investment situation.
2. Generalize the structure of polynomials in order to see similarities to the structure of base-10 representations of integers.

Rigor

- Students build **conceptual understanding** of how polynomials are a system analogous to the integers.
- Students **apply** polynomials to model a simple investment situation.

Coherence

Today

Students identify parallels between the base-10 number system and polynomials and practice evaluating polynomials for different inputs. Students then use the structure of the terms of a polynomial to model a simple investment account **(MP7)**.

Previously

In Lesson 1, students modeled the volume of an open-top box with a polynomial, and examined its graph to determine the maximum value and the domain of the function.

Coming Soon

In Lesson 3, students will describe features of polynomials and their graphs.

Standards

Addressing

HSF.IF.A.2
Use function notation, evaluate functions for inputs in their domains, and interpret statements that use function notation in terms of a context.

Also Addressing: **HSA.APR.A.1, HSA.SSE.A.1, HSA.CED.A.2**

Building On	Building Toward
HSF.IF.A.2	HSA.APR.A.1 HSA.SSE.A.1

Pacing Guide

Suggested Total Lesson Time **~50 min**

Warm-up	Activity 1	Activity 2	Summary	Exit Ticket
5 min	15 min	20 min	5 min	5 min
Pairs	Independent	Independent	Whole Class	Independent
MP7	MP7	MP4, MP5	MP4, MP7	MP4
	HSA.APR.A.1, HSA,SSE.A.1	HSF.IF.A.2, HSA.CED.A.2	HSF.IF.A.2, HSA.CED.A.2	HSF.IF.A.2, HSA.CED.A.2

Amps powered by desmos | Activity and Presentation Slides

For a digitally interactive experience of this lesson, log in to Amplify Math at **learning.amplify.com**.

Materials

- Exit Ticket
- Additional Practice
- Anchor Chart PDF, *Sentence Stems, Notice and Wonder*
- graphing technology

Math Language Development

New words

- *factored form (of a polynomial)*
- *standard form (of a polynomial)*

Review words

- *coefficient*
- *polynomial*
- *terms*

Amps | Featured Activity

Activity 2
Modeling Investments

Students write a polynomial to model investment into a simple savings account. As they change the coefficient and exponent of each term, they observe how the graph changes, and can be used to determine an interest rate for a given account balance.

Amps POWERED BY desmos

Building Math Identity and Community

Connecting to Mathematical Practices

Students might doubt their ability to determine the pattern in how a dot diagram represents an integer and/or a polynomial, and, therefore, lose the self-confidence to approach the task. Explain to students that they will look closely to discern a pattern or structure **(MP7)**. As students work to construct dot diagrams to represent the polynomials, encourage them to regulate against impulses to give up when the structure is not immediately apparent.

Modifications to Pacing

You may want to consider this additional modification if you are short on time.

- In **Activity 2**, Problem 4 may be omitted.

MP7

Warm-up Notice and Wonder

Students examine multiple representations of an integer, as they prepare to see the connections between digits and coefficients, and between place value and exponents.

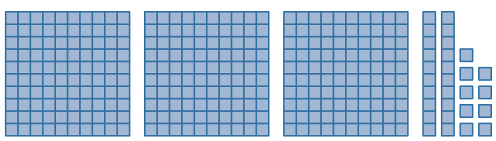
Unit 2 | Lesson 2

Evaluating Polynomials

Let's look at some other situations that polynomials can model.

Warm-up Notice and Wonder

Four ways to represent an integer are shown below. What do you notice? What do you wonder?

Representation A	
Representation B	$300 + 20 + 9$
Representation C	three 100s, two 10s, nine 1s
Representation D	$3(10^2) + 2(10^1) + 9(10^0)$

1. I notice . . .
   Sample responses:
   - Each model represents 329.
   - In Representation A, each of the blocks equals 1 unit.
   - Each model has 3 terms, or sizes, or amounts added together.

2. I wonder . . .
   Sample responses:
   - Can non-integers be represented using the model in Representation D?
   - Can every integer be represented by these models?
   - Why are all the representations in powers of 10?

Log in to Amplify Math to complete this lesson online.

1 Launch

Display all four representations. Then conduct the ***Notice and Wonder*** routine and display the corresponding Anchor Chart PDF.

2 Monitor

Help students get started by asking, "How do the representations relate to one another?"

Look for points of confusion:

- **Thinking Representation D is unrelated to Representation A.** Ask students, "Where do you see powers of 10 in the blocks?"

Look for productive strategies:

- Changing Representations C and D to match Representation B, to confirm they all represent 329.

3 Connect

Have individual students share what they noticed and wondered. Encourage students to point out seemingly contradicting information, such as noting that there are three terms in Representations B and D, but seven groups of blocks in Representation A **(MP7)**.

Highlight that the structure of Representation D is similar to the structure of a quadratic expression, with a 10 instead of an x. All numbers can be represented the way 329 is in this model, using different powers of 10.

Ask:

- "What number would the squares represent if each small square in Representation A was actually 0.1 instead of 1?" 32.9
- "How could that number be written in the three ways shown in Representations B, C, and D?" $30 + 2 + 0.9$; three 10s, two 1s, nine 0.1s; $3(10^1) + 2(10^0) + 9(10^{-1})$

Math Language Development

MLR5: Co-craft Questions

During the Launch, display the four ways to represent an integer. Have students work with their partner to write 2–3 mathematical questions they could ask about the different representations.

Sample questions shown.

- Can non-integers be represented using the model in Representation D?
- Can every integer be represented by these models?
- Why are all the representations in powers of 10?

English Learners: To support students in developing metalinguistic awareness, model how to craft a mathematical question. Consider displaying one of the sample questions.

Power-up

To power up students' ability to write integers using a power of 10, have students complete:

1. 10 can be rewritten using a power of 10, or 10^1. Rewrite 100, 1,000, and 010,00 as a power of 10. 10^2, 10^3, and 10^4
2. 40 can be rewritten as $4(10^1)$. Rewrite 400, 5,000, and 60,000 as a power of 10. $4(10^2)$, $5(10^3)$, and $6(10^4)$

Use: Before the Warm-up

Informed by: Performance on Lesson 1, Practice Problem 6

Independent | 15 min

Activity 1 Polynomials in the Integers

MP7
HSA.APR.A.1, HSA,SSE.A.1

Students evaluate polynomial expressions, while reinforcing their similarity to base-10 representations of integers **(MP7).**

Name: ______ Date: ______ Period: ______

Activity 1 Polynomials in the Integers

Mathematician and educator, James Tanton, helped popularize a representation of integers using a visual dot diagram called *Exploding Dots*. A dot diagram representing 2,463 is shown.

1,000	100	10	1
●●	●●●●	●●●●●●	●●●

1. How is the dot diagram created using the digits that make 2,463?

Sample response: Each column of the diagram represents a place value of the integer. The number of dots in each column represents the digit of that value's place. For example, there are 2 dots in the column labeled 1,000 because there is a 2 in the 1,000 place in 2,463.

2. Construct a dot diagram to represent the integers shown.

a. 3,691

1,000	100	10	1
●●●	●●●●●●	●●●●●●●●●	●

b. 68,204

10,000	1,000	100	10	1
●●●●●●	●●●●●●●●	●●		●●●●

Polynomials can also be represented by a dot diagram. The dot diagram of a polynomial is shown.

x^3	x^2	x	1
●●	●●●●	●●●●●●	●●●

3. Write the polynomial that you think is represented by the dot diagram.

$2x^3 + 4x^2 + 6x + 3$

4. Construct a dot diagram to represent the polynomials shown.

a. $5x^3 + 7x^2 + x + 4$

x^3	x^2	x	1
●●●●●	●●●●●●●	●	●●●●

b. $2x^4 + 5x^2 + 6x + 1$

x^4	x^3	x^2	x	1
●●		●●●●●	●●●●●●	●

1 Launch

Review the dot diagram for 2,463 together as a class. Have students complete Problem 1, and then pause for students to share their thinking.

2 Monitor

Help students get started by asking, "Why are the values of 1,000, 100, 10, and 1 written across the top of the box?"

Look for points of confusion:

- **Misunderstanding how a missing term, with a coefficient of zero, is represented in the dot diagram.** Ask students, "What is the coefficient of a term if it is not written? How is a zero represented in a dot diagram for integers?"

Look for productive strategies:

- Rewriting each integer as a product of an integer and and a power of 10 in Problem 5.

Activity 1 continued >

Differentiated Support

Accessibility: *Guide Processing and Visualization*

Demonstrate for students how the dot diagram is created by displaying a copy of the dot diagram and annotating it with the coefficients for the 1000 and 100 boxes. Then ask, "What number is represented in the 10 and 1 box?"

Math Language Development

MLR3: Critique, Correct, Clarify

Before students share their response for Problem 6, present the following ambiguous conclusion, "This diagram shows that I can use base powers and replace the numbers with variables to find the answer." Ask:

- ***Critique:*** "Do you agree or disagree with this conclusion? Why or why not?"
- ***Correct:*** "Write a clearer statement that is fully true."
- ***Clarify:*** "What do you think the person that wrote the statement misunderstood?"

English Learners: Allow students time to rehearse what they will say with a partner before sharing with the whole class.

Independent | 15 min

Activity 1 Polynomials in the Integers (continued)

MP7
HSA.APR.A.1, HSA,SSE.A.1

Students evaluate polynomial expressions, while reinforcing their similarity to base-10 representations of integers **(MP7).**

Activity 1 Polynomials in the Integers (continued)

Consider the polynomial function $p(x) = 5x^3 + 6x^2 + 4x$.

5. Evaluate the function at $x = 10$ by computing each term. Show your thinking.

$p(10) = \mathbf{5(10^3)} + \mathbf{6(10^2)} + \mathbf{4(10)}$

$= \mathbf{5{,}000} + \mathbf{600} + \mathbf{40}$

$= \mathbf{5{,}640}$

6. In the dot diagrams shown, represent the function $p(x)$ and the integer you determined for $p(10)$.

$p(x)$				$p(10)$			
x^3	x^2	x	1	1,000	100	10	1
•••••	••••••	••••		•••••	••••••	••••	

7. What is the solution to $5x^3 + 6x^2 + 4x = 5640$? Explain your thinking.

10; Sample response: $p(10) = 5640$, which confirms that the solution to the equation $5x^3 + 6x^2 + 4x = 5640$ is $x = 10$.

8. In the function $p(x) = 5x^3 + 6x^2 + 4x$, x can represent a number other than 10. Evaluate the function at $x = -5$ and $x = 15$. Show your thinking.

$p(-5) = -495$;
$p(-5) = 5(-5)^3 + 6(-5)^2 + 4(-5)$
$= (-625) + 150 + (-20)$
$= -495$

$p(15) = 18285$;
$p(15) = 5(15)^3 + 6(15)^2 + 4(15)$
$= 16875 + 1350 + 60$
$= 18,285$

Featured Mathematician

James Tanton

James Tanton is a mathematician and educator. Along with the awards for teaching excellence he has received from St. Mary's College and Princeton University, he is known for creating an interactive method for representing integers and polynomials called *Exploding Dots*. Exploding Dots illustrates how arithmetic, polynomial algebra, and infinite sums are connected by the concepts of place value and substitution.

Courtesy of James Tanton.

3 Connect

Have individual students share their response to Problem 6.

Highlight that polynomials can represent decimal forms of numbers when the variable is 10. Polynomials also act a lot like the number system in that we can add, subtract, and multiply polynomials to get other polynomials, which students will see in the next few lessons.

Ask, "The polynomials in this activity are in standard form. What do you think is the difference between the standard and factored forms of polynomials?" Standard form of a polynomial is fully expanded with no parentheses. Factored form of a polynomial is written as a product of expressions.

Define ***standard form (of a polynomial)*** as an expression that is fully expanded with no parentheses, with terms combined and written in descending order, that is from the highest to the lowest exponent, and ***factored form (of a polynomial)*** is written as a product of expressions.

Featured Mathematician

James Tanton

Have students read about mathematician and educator James Tanton. His "Exploding Dots" diagrams show us how arithmetic, polynomial algebra, and infinite sums are elegantly represented and connected to one another.

Activity 2 A Yearly Gift

MP4, MP5
HSF.IF.A.2, HSA.CED.A.2

Students examine the structure of the terms of a polynomial and write a polynomial to model a savings scenario **(MP4).**

Amps Featured Activity Modeling Investments

Name: ____________ Date: ________ Period: ________

Activity 2 A Yearly Gift

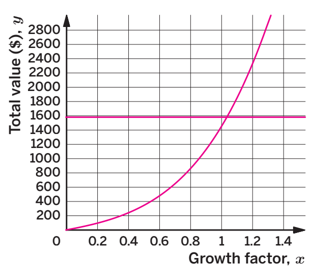
At the end of high school, Clare's aunt started saving money for her to use after college graduation. She made an initial deposit of \$300 into a savings account. If r represents the annual interest rate of the account, then at the end of each school year, the balance in the account is multiplied by the growth factor $x = 1 + r$.

1. After 1 year, the total value of the account is $300x$. After 2 years, the total value of the account is $300x \cdot x = 300x^2$. Write an expression for the total value of the account after 4 years in terms of x.
   $300x \cdot x \cdot x \cdot x$, or $300x^4$

2. If Clare's aunt had invested another \$500 at the end of Clare's first year of college, what would the total value of the account be after 4 years? Write an expression to represent the account's total value in terms of x.
   $300x^4 + 500x^3$

3. Suppose that \$250 was invested at the end of Clare's second year of college, and \$400 at the end of her third year of college, in addition to the initial \$300 and the \$500 invested at the end of her first year. Write an expression that represents the total value of the account after graduation — 4 years — in terms of x.
   $300x^4 + 500x^3 + 250x^2 + 400x$

4. The total value y, in dollars, after four years is a function $y = C(x)$ of the growth factor x. If the total Clare receives after graduation is represented by $C(x) = 1{,}580$, sketch a graph to approximate the interest rate r of the account. Explain your thinking.
   3.5%; Sample response: Looking at the graph of $y = 300x^4 + 500x^3 + 250x^2 + 400x$ and $y = 1{,}580$, the two graphs intersect at approximately 1.035. If $x = 1 + r$, then r is 0.035, or 3.5%. So, with an interest rate of 3.5%, the account will grow to \$1,580 in 4 years.



5. How might the polynomial function change if Clare's aunt invested the same amounts each year, but made these investments 3 years ago, rather than 4 years ago? Write the polynomial function $T(x)$ to represent this new scenario. Explain your thinking.
   $T(x) = 300x^3 + 500x^2 + 250x + 400$; Sample response: The exponent on each term would decrease by 1 because there would be 1 less year of investing.

STOP

1 Launch

Arrange students in pairs. Review the narrative together as a class. Provide graphing technology. Have students complete Problems 1 and 2 before checking in with their partner, and then complete the rest of the activity independently.

2 Monitor

Help students get started by asking, "Why does x have an exponent of 2 in the expression $300x \cdot x = 300x^2$ in Problem 1?"

Look for points of confusion:

- **Misunderstanding the growth rate $x = 1 + r$.** Have students assume a specific value for r and consider what steps they must take to calculate the amount in the account after a full year to see why using x is beneficial.

Look for productive strategies:

- Choosing a value of r, and substituting the corresponding value of x into their polynomials to check their accuracy.

3 Connect

Have individual students share their strategy for writing the polynomials to model the total value of the savings **(MP5)**.

Highlight that the exponent of each term in the polynomial represents the number of years that have passed since the amount was invested. The terms are determined by the amount invested and the number of years that have passed.

Display the graph of $C(x)$ and $y = 1580$. Ask students how they determined the axes limits to identify the intersection point.

Ask, "How might the polynomial change if Clare's aunt invested along the same timeline, but she invested different amounts each year?" The coefficients of the polynomial would change to represent the amounts she invested each year.

Differentiated Support

Accessibility: *Optimize Access to Technology*

Have students use the Amps slides for this activity, in which they can write a polynomial to model investment into a simple savings account.

Extension: *Math Enrichment*

During the Connect, after you ask students how the polynomial might change if Clare's aunt invested along the same timeline but invested different amounts each year, provide students with examples of different amounts and ask them to create a polynomial with the different amounts provided.

Responses may vary.

Math Language Development

MLR6: Three Reads

Use this routine to help students make sense of the introductory text and problem prompts.

- **Read 1:** Students understand that Clare's aunt is investing money.
- **Read 2:** Listen for, and amplify, the important quantities that vary in relation to each other in this situation: number of years, growth factor, total value, and the fact that the additional investments are at the same annual interest rate.
- **Read 3:** Have students read the problem prompts and ask students to brainstorm strategies for how they will complete the first problem.

English Learners: Annotate and highlight that in the questions, the additional investments are made at the same annual interest rate.

Whole Class | 5 min

MP4, MP7
HSF.IF.A.2, HSA.CED.A.2

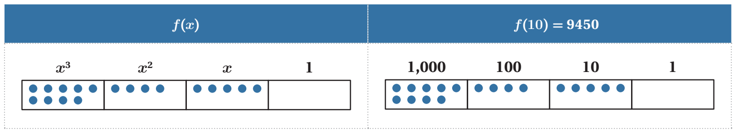
Summary

Review and synthesize how polynomials are similar to integers and can be used to model situations.

Summary

In today's lesson . . .

You determined that polynomials are a system similar to integers. For example, the terms of the function $f(x) = 9x^3 + 4x^2 + 5x$ can be used to represent the integer 9,450 when the function is evaluated at $x = 10$, because $f(10) = 9(10^3) + 4(10^2) + 5(10^1) = 9000 + 400 + 50 = 9450$. This similarity between polynomials and integers can be represented by dot diagrams.



Polynomial functions can model situations, such as money invested over the lifetime of an account. For example, suppose you have an account that was opened 5 years ago with an initial investment of \$200. 3 years ago \$100 was invested, and 1 year ago \$50 was invested. If the annual interest rate is r, then the growth factor is $x = 1 + r$. We can represent the current amount in the account in terms of x using the polynomial function $C(x) = 200x^5 + 100x^3 + 50x$.

Polynomials can be written in different forms, just like quadratic expressions. The ***standard form of a polynomial*** is fully expanded with no parentheses. The ***factored form of a polynomial*** is written as a product of expressions.

Reflect:

Synthesize

Display the polynomial functions $f(x) = 9x^3 + 4x^2 + 5x$ and $C(x) = 200x^5 + 100x^3 + 50x$.

Have students share how $f(x)$ can be used to represent the integer 9,450, when x equals 10 **(MP7)**.

Formalize vocabulary:

- *factored form (of a polynomial)*
- *standard form (of a polynomial)*

Highlight that polynomials are a system similar to integers. Integers can be represented using powers of 10, just as polynomials use powers of x. The digits of an integer written in base 10 correspond to a polynomial's coefficients. Polynomials can be used to model a variety of situations as well. Each term of the polynomial reveals key features of the situations it models **(MP4)**.

Ask:

- "If $C(x)$ models money invested in a savings account that earns interest every year, how long ago did the account open? How can you tell?" \$200 has earned interest 5 times, so the account was opened 5 years ago.
- "What do the coefficients and the exponents in $C(x)$ represent?" The coefficients represent the amount of money invested 1, 3, and 5 years ago. The exponents represent the number of years ago each amount was invested.

Reflect

After synthesizing the concepts of the lesson, allow students a few moments for reflection on one of the Essential Questions for this unit. Encourage them to record any notes in the *Reflect* space provided in the Student Edition. To help them engage in meaningful reflection, consider asking:

- "How are polynomials similar to integer expressions written with powers of 10?"

Math Language Development

MLR2: Collect and Display

As students formalize the new vocabulary for this lesson, ask them to refer to the class display that you started in this unit. Ask them to review and reflect on the terms and phrases related to the terms *factored form* and *standard form (of a polynomial)* that were added to the display during the lesson.

Independent | 5 min
MP4
HSF.IF.A.2, HSA.CED.A.2

Exit Ticket

Students demonstrate their understanding by evaluating a polynomial in context, while also changing the polynomial to model a scenario **(MP4)**.

Printable

Name: Date: Period:

Exit Ticket 2.02

A savings account has an annual interest rate of r, which means it grows by a factor of $x = 1 + r$ each year. The function $A(x) = 800x^4 + 350x^3 + 500x^2 + 600x$ represents the amount in the account after 4 years.

1. What is the total amount in the account if the interest rate for the account is 3% each year? Show or explain your thinking.
   Approximately \$2,431;
   $x = 1 + 0.03,\ A(1.03) = 800(1.03)^4 + 350(1.03)^3 + 500(1.03)^2 + 600(1.03) = 2431$

2. How much money was initially deposited into the account?
   \$800

3. After 5 years, another \$200 is added to the account. Write a new function $B(x)$ that represents how much is in the account after 5 years.
   $B(x) = 800x^5 + 350x^4 + 500x^3 + 600x^2 + 200x$

Self-Assess

1 I don't really get it | 2 I'm starting to get it | 3 I got it

a. I can use polynomials to model and understand different types of scenarios. 1 2 3

b. I can understand how polynomials form a system similar to integers. 1 2 3

c. I can evaluate polynomials. 1 2 3

Success looks like . . .

- **Goal:** Creating a polynomial to model a simple investment situation.
  - ✓ Writing and evaluating a polynomial to model a simple savings account, given the number of years and amount invested.
- **Goal:** Generalizing the structure of polynomials in order to see similarities to the structure of base-10 representations of integers.

Suggested next steps

If students inaccurately evaluate the polynomial in Problem 1, consider:

- Reviewing evaluating polynomials from Activity 2.
- Assigning Practice Problems 2 and 3.
- Asking, "How do we determine the value of x using the interest rate?"

If students inaccurately identify the initial amount in Problem 2, consider:

- Reviewing the structure of the polynomial used to represent the investment from Activity 2.
- Assigning Practice Problems 2 and 3.
- Asking, "How long ago was the account opened? How do you know?"

If students inaccurately write the polynomial in Problem 3, consider:

- Reviewing the meaning of the coefficients and exponents in the polynomial that models the investment from Activity 2.
- Assigning Practice Problems 2 and 3.
- Asking, "What does each term in the original polynomial $A(x)$ tell us? How can you use this structure to write a new polynomial that models this scenario?"

Professional Learning

This professional learning moment is designed to be completed independently or collaboratively with your fellow mathematics educators. Prompts are provided so that you can reflect on this lesson before moving on to the next lesson.

Points to Ponder . . .

- What worked and didn't work today? How did the Notice and Wonder support students in understanding how polynomials are a system similar to integers?
- What challenges did students encounter as they worked on modeling the investment scenario with a polynomial? How did they work through them? What might you change the next time you teach this lesson?

Practice

Independent

Name: Date: Period:

Practice

1. Select *all* polynomial expressions equivalent to $6x^4 + 4x^3 - 7x^2 + 5x + 8$.
   - A. $16x^{10}$
   - B. $-7x^3 + 6x^5 + 4x^4 + 5x^2 + 8x$
   - C. $6x^4 - 7x^2 + 5x + 8 + 4x^3$
   - D. $8 + 5x + 7x^2 - 4x^3 + 6x^4$
   - E. $8 + 5x - 7x^2 + 4x^3 + 6x^4$

2. Each year, a certain amount of money is deposited into an account with an annual interest rate of r, so that the balance in the account is multiplied by a growth factor of $x = 1 + r$ by the end of each year. \$500 is deposited at the start of the first year, an additional \$200 is deposited at the start of the next year, and \$600 at the start of the following year.
   - a. Write a function $f(x)$ for the value of the account at the end of 3 years, in terms of the growth factor x.
     $f(x) = 500x^3 + 200x^2 + 600x$
   - b. What is the amount (to the nearest cent) in the account at the end of 3 years if the interest rate is 2%? Show your thinking.
     \$1,350.68; $x = 1 + 0.02$;
     $f(1.02) = 500\,(1.02)^3 + 200\,(1.02)^2 + 600\,(1.02)$
     $f(1.02) \approx 530.60 + 208.08 + 612$
     $f(1.02) \approx 1350.68$

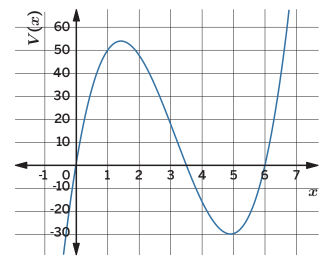
3. Consider the polynomial function $p(x) = 5x^3 + 8x^2 - 3x + 1$. Evaluate the function for the following values of x.
   - a. $x = 0$
     $p(0) = 1$
   - b. $x = 2$
     $p(2) = 67$
   - c. $x = -2$
     $p(-2) = -1$

Practice

Name: Date: Period:

4. A fence is being installed around a rectangular playground by using the wall of a daycare building for one side and 200 m of fencing for the other three sides. The area $A(x)$ in square meters, of the playground space is a function of the length x, in meters, of each of the sides perpendicular to the wall of the daycare building.
   - a. What is the area of the playground when $x = 50$? Show your thinking.
     5,000 m²; Sample response: Area $= l \cdot w = (50)(200 - 2(50)) = 5000$
   - b. Write an expression for the function $A(x)$.
     $(x)(200 - 2x)$
   - c. What is a reasonable domain for function A in this context? Explain your thinking.
     Sample response: $0 < x < 100$; One side of the fence must be less than 100 m because the total amount of fencing is only 200 m. With a rectangular playground, the side opposite the building would need to be several meters long. It does not make sense for a playground to be long and narrow.

5. Tyler determines an expression for $V(x)$ that represents the volume of an open-top box, in cubic inches, in terms of the length x, in inches, of the square corner cutouts used to make it from a sheet of cardboard. When Tyler allows x to take on any value between -1 and 7, the graph shown is produced.



   - a. What would be a more appropriate domain for Tyler to use instead?
     $0 < x < 3.5$
   - b. What is the approximate maximum volume of the box?
     55 in³

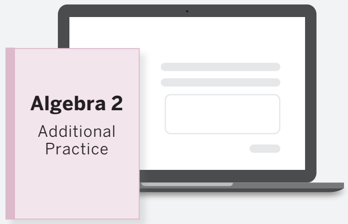
6. Label the following functions as *exponential*, *quadratic*, *linear*, or *none of the above*.
   - a. $f(x) = x(4 - x)$
     Quadratic
   - b. $g(x) = 3x^2 + 5x - 3(x + x^2)$
     Linear
   - c. $h(x) = 4 \cdot 5^x$
     Exponential
   - d. $k(x) = 5x(x^2 + 7)$
     None of the above

Practice Problem Analysis

Type	Problem	Refer to	Standard(s)	DOK
On-lesson	1	Activity 1	HSA.APR.A.1	1
	2	Activity 2	HSF.IF.A.2, HSA.CED.A.2	2
	3	Activity 1	HSF.IF.A.2	1
Spiral	4	Unit 2 Lesson 1	HSA.CED.A.2	2
	5	Unit 2 Lesson 1	HSF.IF.B.4, HSF.IF.B.5	2
Formative	6	Unit 2 Lesson 3	HSA.SSE.A	2

Power-up: If students need additional support with the key prerequisite concept or skill this problem addresses, consider assigning the Power-up in the next lesson.

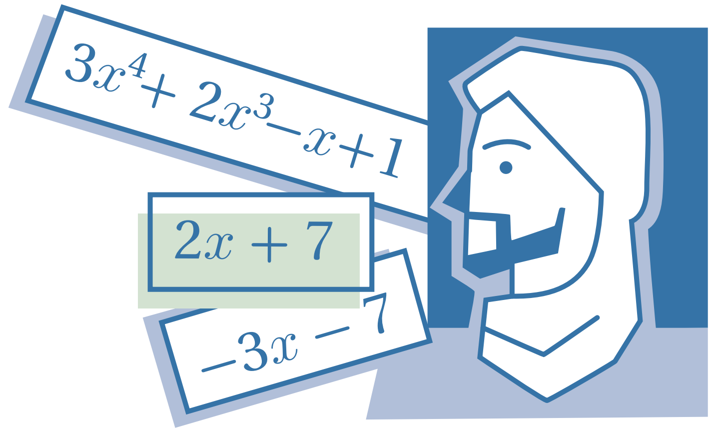
Additional Practice Available



For students who need additional practice in this lesson, assign the **Algebra 2 Additional Practice**.

Adding Polynomials

Let's do some arithmetic with polynomials.



Focus

Goals

1. Comprehend that when polynomials are added or subtracted, the result is a polynomial.
2. **Language Goal:** Justify conclusions about what happens when integers or polynomials are combined using arithmetic operations. **(Speaking and Listening)**

Coherence

Today

Students are introduced to the idea of closure (although they will not formally use this term in this course), and that both integers and polynomials are closed under addition and subtraction. Students practice adding and subtracting polynomials, strengthen their understanding of polynomials, and note similarities between integers and polynomials **(MP3)**.

Previously

In Lesson 3, students identified the degree and leading coefficient of a polynomial and connected these features to the shape of its graph.

Coming Soon

In Lesson 5, students will multiply polynomials and make conjectures about how the degree of a product relates to the degrees of its factors.

Rigor

- Students build **conceptual understanding** of how polynomials form a system analogous to the integers.
- Students develop **procedural fluency** adding and subtracting polynomials.

Standards

Addressing

HSA.APR.A.1
Understand that polynomials form a system analogous to the integers, namely, they are closed under the operations of addition, subtraction, and multiplication; **add, subtract**, and multiply **polynomials**.

Also Addressing: **HSA.APR.A**

Building On	Building Toward
HSA.SSE.A	HSA.APR.A.1

Pacing Guide

Suggested Total Lesson Time **~50 min**

Warm-up	Activity 1	Activity 2	Summary	Exit Ticket
5 min	15 min	20 min	5 min	5 min
Whole Class	Pairs	Pairs	Whole Class	Independent
MP3, MP6	MP3	MP6	MP6	MP6
HSA.APR.A.1	HSA.APR.A.1, HSA.APR.A	HSA.APR.A.1, HSA.APR.A	HSA.APR.A.1, HSA.APR.A	HSA.APR.A.1, HSA.APR.A

Amps powered by desmos | **Activity and Presentation Slides**

For a digitally interactive experience of this lesson, log in to Amplify Math at **learning.amplify.com**.

Practice Independent

Materials

- Exit Ticket
- Additional Practice
- Activity 2 PDF, pre-cut cards, one set per pair

Math Language Development

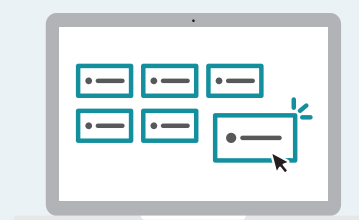
Review words

- *coefficient*
- *degree*
- *leading coefficient*
- *polynomial*
- *standard form (of a polynomial)*
- *terms*

Amps | **Featured Activity**

Activity 2
Polynomial Puzzle

Students determine equivalent polynomial expressions by arranging cards with polynomials written on each side.



Building Math Identity and Community

Connecting to Mathematical Practices

Students might struggle with accurately determining the difference between two polynomials and make mathematical errors. Explain to students that it is important for them to justify their conclusions and communicate them to their partner **(MP3)** when explaining their reasoning about the sum and difference of two polynomials. Remind students to listen actively to their partner's reasoning in order to interpret whether or not they agree with the explanation.

Modifications to Pacing

You may want to consider this additional modification if you are short on time.

- In **Activity 2**, provide students with the arrangement of the first row of cards and have them arrange the remaining cards.

Whole Class | 5 min

Warm-up True or False?

MP3, MP6
HSA.APR.A.1

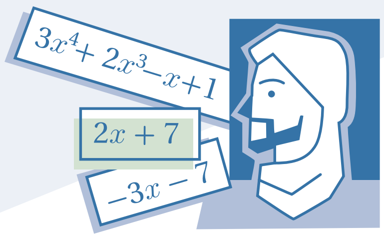
Students determine the validity of statements to acknowledge the closure of the integers under addition and subtraction **(MP6).**

Name: Date: Period:

Unit 2 | Lesson 4

Adding Polynomials

Let's do some arithmetic with polynomials.



Warm-up True or False?

Determine whether each statement is true or false. Explain your thinking.

1. The sum of two even numbers is always an even number.
   True. Sample response: The sum of a multiple of 2 and another multiple of 2 is a multiple of 2, which is even.
2. The difference between two even numbers is always an even number.
   True. Sample response: The difference of a multiple of 2 and another multiple of 2 is a multiple of 2.
3. The sum of two odd numbers is always an odd number.
   False. Sample response: The sum of two odd numbers is an even number. For example: $3 + 5 = 8$
4. The difference between two odd numbers is always an odd number.
   False. Sample response: The difference of two odd numbers is an even number. For example: $11 - 7 = 4$

Log in to Amplify Math to complete this lesson online.

1 Launch

Display each statement, one at a time. Give students a minute of think-time, then use the ***True or False*** routine for each statement before they write their reasoning.

2 Monitor

Help students get started by providing them with integers or an expression that aligns to each problem's criteria.

Look for points of confusion:

- **Using one example to make a generalization.** Ask, "How can you be sure that your conclusion is true for all even (or odd) numbers?"

Look for productive strategies:

- Using expressions such as, $2n$ and $2n + 1$ to represent even or odd numbers.

3 Connect

Have individual students share their thinking for each problem.

Highlight that some operations on a type of number will produce numbers of that same type, but others will not. For example, performing multiplication on odd numbers always produces an odd number, but performing addition or subtraction on odd numbers will not produce an odd number **(MP3)**.

Ask, "If you have several integers, what operation(s) could you perform with them to result in something that is *not* an integer?" Sample response: Divide one of them by the other, or average them.

Math Language Development

MLR8: Discussion Supports—Press For Reasoning

During the Connect, as students share their thinking for each problem, press for details in their reasoning. For example:

If a student says . . .	Press for details by asking . . .
"I think two even numbers always add to an even number."	"How did you decide that? Can you draw a diagram that shows why this is always true?"

Power-up

To power up students' ability to add and subtract polynomials, ask:

1. Combine terms to determine the sum $4x + 3x - x$. $6x$
2. How can you use coefficients and exponents to add polynomials?
   Terms that have the same power of x can be added by adding together the coefficients of these terms. The power of x remains the same.
3. Can the terms $5x^3 - x^2$ be combined? Explain your thinking.
   No. The terms have different powers of x so they cannot be subtracted.

Use: Before Activity 1

Informed by: Performance on Lesson 3, Practice Problem 6

Pairs | 15 min

Activity 1 Experimenting With Polynomials

MP3
HSA.APR.A.1, HSA.APR.A

Students add and subtract polynomials to make arguments about the closure of polynomials under these operations **(MP3).**

Activity 1 Experimenting With Polynomials

The Persian mathematician, al-Karajī, examined the arithmetic of polynomials, forming rules for these operations. Study some of these rules by analyzing the four polynomials shown.

Polynomial A	Polynomial B	Polynomial C	Polynomial D
$3x^4 + 2x^3 - x + 1$	$-5x^4 - x^3 + 6x + 9$	$2x + 7$	$-3x - 7$

Work with your partner to complete Columns 1 and 2.

Column 1

1. Add the following polynomials:
   a. Polynomial A + Polynomial B
   $-2x^4 + x^3 + 5x + 10$
   b. Polynomial A + Polynomial C
   $3x^4 + 2x^3 + x + 8$
   c. Polynomial B + Polynomial D
   $-5x^4 - x^3 + 3x + 2$
2. Is the *sum* of two polynomials always a polynomial? Explain your thinking.
   Yes. Sample response: The terms will still be a sum of multiples of powers of x.

Column 2

3. Subtract the following polynomials:
   a. Polynomial B – Polynomial A
   $-8x^4 - 3x^3 + 7x + 8$
   b. Polynomial A – Polynomial D
   $3x^4 + 2x^3 + 2x + 8$
   c. Polynomial D – Polynomial C
   $-5x - 14$
4. Is the *difference* between two polynomials always a polynomial? Explain your thinking.
   Yes. Sample response: The terms will still be a sum of multiples of powers of x.

5. Do you think multiplying or dividing polynomials always results in a polynomial? Explain your thinking.
   Sample response: Yes. Multiplying or dividing a pair of terms still results in a term that is a multiple of powers of x. (While mathematically incorrect, this is an acceptable response at this point in the unit.)

Featured Mathematician

Abū Bakr Muḥammad ibn al Ḥasan al-Karajī

Al-Karajī was a 10th-century Persian mathematician and engineer who systematically studied the algebra of exponents. Through his work with algebra and polynomials, he discovered and developed rules for arithmetic operations for adding, subtracting, and multiplying polynomials.

1 Launch

Arrange students in pairs. Assign partners to either Column 1 or 2. Have partners complete the problems in their columns together, as well as Problem 5.

2 Monitor

Help students get started by reviewing how to combine terms in a polynomial.

Look for points of confusion:

- **Misunderstanding the definition of a polynomial.** Have students define the term *polynomial*. Remind them that polynomials are the sum of terms that are multiples of powers of x.

Look for productive strategies:

- Multiplying and dividing one-term polynomials (i.e., monomials) to determine their response for Problem 5.

3 Connect

Have pairs of students share their responses to Problems 2 and 4.

Highlight that the sum and difference of two polynomials is always a polynomial. Terms with the same variable and exponent are combined, and the result is a term that is still a multiple of a power of x. Note that constants are technically polynomials (with degree 0), so a polynomial subtracted from itself is still a polynomial.

Ask, "How might showing that the sum and difference of integers always results in an integer help to show that this is also true for polynomials?"
Integers can be written as an expression that is a sum of powers of 10, just as polynomials are expressions that are the sum of powers of x.

Differentiated Support

Accessibility: *Activate Prior Knowledge*

Reminds students that they have learned about combining like terms in prior grades and that skill is required for finding the sum and difference of polynomial expressions.

Math Language Development

MLR8: Discussion Supports—Revoicing

During the Connect, as students share their responses to Problems 2 and 4, revoice their ideas in the form of a question using appropriate mathematical language or language from the context. For example:

If a student says . . .	Revoice their ideas by asking . . .
"I added two polynomials together and found that the sum is still a polynomial."	"By adding the terms of the polynomials, did you notice that the terms will still be a sum of multiples of powers of x?"

Featured Mathematician

Abū Bakr Muḥammad ibn al Ḥasan al-Karajī

Have students read about al-Karajī, a 10th-century Persian mathematician and engineer whose work on algebra and polynomials led to the development of the rules for arithmetic operations for adding, subtracting, and multiplying polynomials.

Pairs | 20 min

Activity 2 Polynomial Puzzle

MP6
HSA.APR.A.1, HSA.APR.A

Students match equivalent polynomial expressions to practice adding and subtracting polynomials **(MP6).**

Amps Featured Activity Polynomial Puzzle

Name: ____________ Date: ____________ Period: ____________

Activity 2 Polynomial Puzzle

You will be given a set of cards. The cards can be arranged to form a square that is three blocks wide and 3 blocks long. Arrange the cards so that equivalent polynomials are on adjacent sides. Show the order of the cards by writing the letter names of the cards in each square.

A	E	D
B	I	H
F	C	D

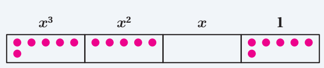
Compare and Connect: Your teacher will display two questions about how you arranged the cards. Be ready to make comparisons between your classmates' strategies.

Are you ready for more?

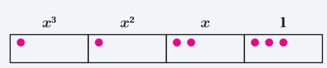
Recall that you can represent polynomials with dot diagrams. Let's see how these diagrams can determine the sum of $(6x^3 + 5x^2 + 6) + (x^3 + x^2 + 2x + 3)$.

1. Represent each polynomial with a dot diagram.

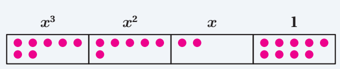
$6x^3 + 5x^2 + 6$



$x^3 + x^2 + 2x + 3$



2. Using the dot diagrams of polynomials, complete the dot diagram to represent the sum of $(6x^3 + 5x^2 + 6) + (x^3 + x^2 + 2x + 3)$, and then write the polynomial it represents.



$7x^3 + 6x^2 + 2x + 9$

3. How can you use dot diagrams to add polynomials?
Sample response: I can place dots to represent each term of the polynomials in the corresponding column for each term. Then I can add together these dots to determine the term of the polynomial that represents the sum.

STOP

1 Launch

Students remain in pairs. Distribute a set of the pre-cut Activity 2 PDF cards to each pair of students. Review the activity together as a class, and give an example of how to match two sides.

2 Monitor

Help students get started by suggesting they begin with expressions whose terms can be combined, and then search for the equivalent expression on the other cards.

Look for points of confusion:

- **Overlooking equivalent expressions.** Have students write all the polynomials in standard form to help match sides.

Look for productive strategies:

- Confirming all adjacent sides of the cards shows equivalent expressions.

3 Connect

Display the correct arrangement of cards.

Have pairs of students share their strategies for matching cards.

Highlight that it is helpful to write polynomials in standard form to determine whether two polynomials are equivalent.

Ask, "What features of polynomials could you use to efficiently determine whether they are *not* equivalent?"
Sample response: I could compare the leading coefficient and degree of the polynomials after terms have been combined. If these are not equivalent, then the polynomials are not equivalent.

Differentiated Support

Accessibility: *Optimize Access to Technology*

Have students use the Amps slides for this activity, in which they can determine equivalent polynomial expressions by arranging cards with polynomials written on each side.

Math Language Development

MLR7: Compare and Connect

During the Connect, as students share their strategies for matching cards, draw connections between how students rewrote polynomials in standard form to determine equivalency. Ask:

- "How did you use the structure of the polynomials to determine equivalency?"
- "How did you use the structure of the polynomials to determine which polynomials were not equivalent?"

English Learners: Encourage students to refer to and use language from the class display to support their developing math language while sharing their strategies.

Whole Class | 5 min

MP6
HSA.APR.A.1, HSA.APR.A

Summary

Review and synthesize adding and subtracting polynomials and how polynomials are closed under these operations.

Summary

In today's lesson . . .

You recalled that when you add two integers or subtract one from the other, the result is always an integer. The same thing is true for polynomials. Combining polynomials by adding or subtracting them will always result in another polynomial.

When you add and subtract polynomials, it is convenient to combine terms with the same variable and exponent. For example: $5x^2 + 7x^3 - 10x^5 + 2x^3 - x^2 = -10x^5 + 9x^3 + 4x^2$

Reflect:

Synthesize

Display the polynomial equation, that shows which terms are combined:

$$5x^2 + 7x^3 - 10x^5 + 2x^3 - x^2 = -10x^5 + 9x^3 + 4x^2$$

Have students share their strategy for adding and subtracting polynomials **(MP6)**.

Highlight that adding two integers or subtracting one from the other, always results in an integer. The same is true for polynomials: Adding or subtracting polynomials will always result in a polynomial.

Ask, "Can you think of two polynomials whose quotient is not a polynomial?" Sample response: Dividing x by x^2, does not result in a polynomial.

Reflect

After synthesizing the concepts of the lesson, allow students a few moments for reflection. Encourage them to record any notes in the *Reflect* space provided in the Student Edition. To help them engage in meaningful reflection, consider asking:

- "How does showing that adding two integers always produces an integer help to show that adding two polynomials always results in a polynomial?"

Independent | 5 min

MP6
HSA.APR.A.1, HSA.APR.A

Exit Ticket

Students demonstrate their understanding by adding and subtracting polynomials and explain why the sum and difference between two polynomials is a polynomial **(MP6).**

Printable

Name: ______ Date: ______ Period: ______

Exit Ticket 2.04

1. Add or subtract the following polynomials.

   a. $(-9x^2 + 5x) + (x^2 + x + 7)$
   $-8x^2 + 6x + 7$

   b. $(7x^2 - x^4 + 2) - (2x^2 - x - x^4)$
   $5x^2 + x + 2$

2. Explain why the sum of and difference between two polynomials is a polynomial.
   Sample response: When adding or subtracting the terms of polynomials, every term will be a constant times a whole number power of x, so the sum or difference will be a polynomial.

Self-Assess

1 I don't really get it | 2 I'm starting to get it | 3 I got it

a. I understand that adding or subtracting polynomials always results in another polynomial.
1 2 3

b. I can add or subtract polynomials.
1 2 3

Success looks like . . .

» **Goal:** Comprehending that when polynomials are added or subtracted, the result is a polynomial.

✓ Adding and subtracting polynomial expressions in Problem 1.

» **Language Goal:** Justifying conclusions about what happens when integers or polynomials are combined using arithmetic operations. **(Speaking and Listening)**

✓ Explaining why the sum of and difference between two polynomials is a polynomial in Problem 2.

Suggested next steps

If students inaccurately add or subtract the polynomials in Problem 1, consider:

- Reviewing how to add and subtract polynomials and combine terms from Activity 1.
- Assigning Practice Problems 1 and 2.
- Asking, "Why can some terms be combined and other terms cannot?"

If students inaccurately or vaguely explain why polynomials are closed under addition and subtraction in Problem 2, consider:

- Reviewing why the sum and difference of two polynomials is a polynomial, from Activity 1.
- Assigning Practice Problem 3.
- Asking, "What defines a polynomial?"

Professional Learning

This professional learning moment is designed to be completed independently or collaboratively with your fellow mathematics educators. Prompts are provided so that you can reflect on this lesson before moving on to the next lesson.

Points to Ponder . . .

- What worked and didn't work today? How did the ***True or False*** routine support students in explaining why polynomials are closed under addition and subtraction?
- What trends do you see in participation? What might you change the next time you teach this lesson?

Math Language Development

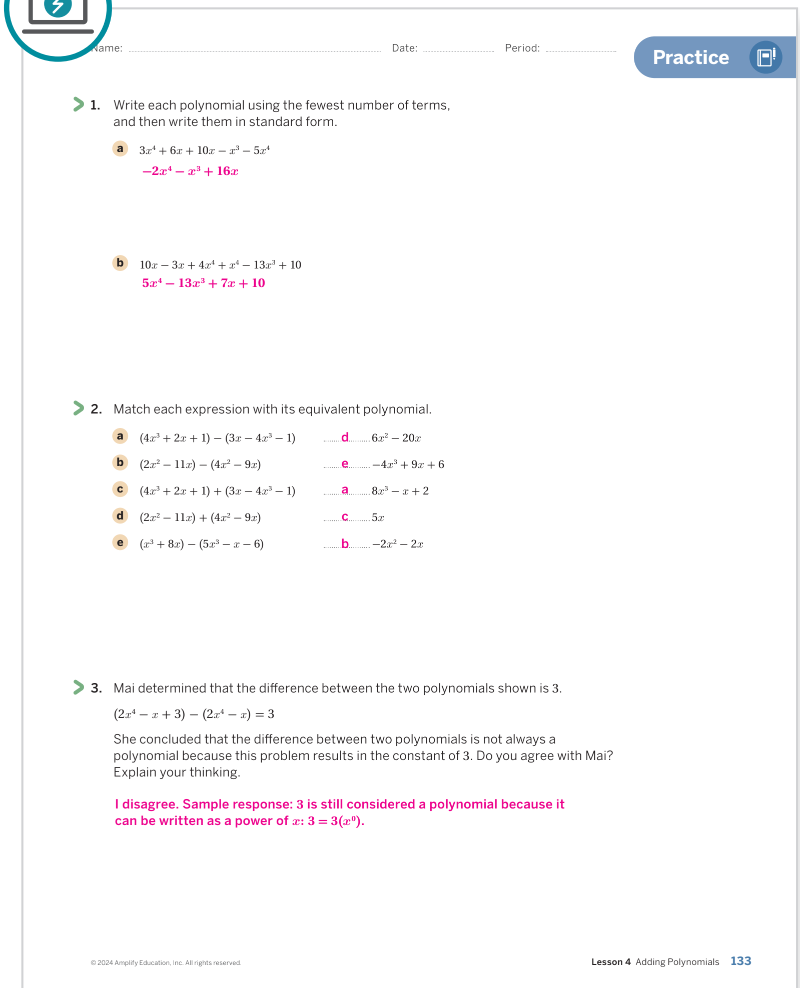
Language Goal: Justifying conclusions about what happens when integers or polynomials are combined using arithmetic operations.

Reflect on students' language development toward this goal.

- How did students begin to justify their conclusions about what happens when polynomials are combined together?
- How did the math language routines during this lesson help students strengthen their justifications?

Practice

Independent



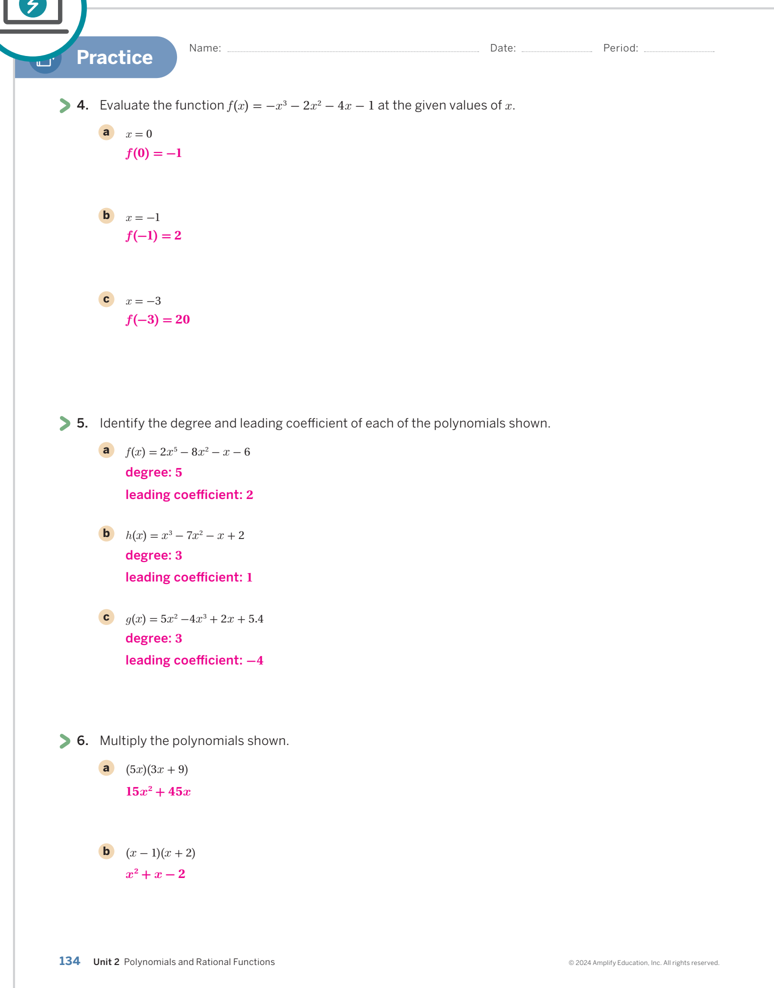
Name: Date: Period:

Practice

1. Write each polynomial using the fewest number of terms, and then write them in standard form.
   - a. $3x^4 + 6x + 10x - x^3 - 5x^4$
     $-2x^4 - x^3 + 16x$
   - b. $10x - 3x + 4x^4 + x^4 - 13x^3 + 10$
     $5x^4 - 13x^3 + 7x + 10$

2. Match each expression with its equivalent polynomial.
   - a. $(4x^3 + 2x + 1) - (3x - 4x^3 - 1)$ — **d** $6x^2 - 20x$
   - b. $(2x^2 - 11x) - (4x^2 - 9x)$ — **e** $-4x^3 + 9x + 6$
   - c. $(4x^3 + 2x + 1) + (3x - 4x^3 - 1)$ — **a** $8x^3 - x + 2$
   - d. $(2x^2 - 11x) + (4x^2 - 9x)$ — **c** $5x$
   - e. $(x^3 + 8x) - (5x^3 - x - 6)$ — **b** $-2x^2 - 2x$

3. Mai determined that the difference between the two polynomials shown is 3.
   $(2x^4 - x + 3) - (2x^4 - x) = 3$
   She concluded that the difference between two polynomials is not always a polynomial because this problem results in the constant of 3. Do you agree with Mai? Explain your thinking.
   I disagree. Sample response: 3 is still considered a polynomial because it can be written as a power of x: $3 = 3(x^0)$.



Practice Name: Date: Period:

4. Evaluate the function $f(x) = -x^3 - 2x^2 - 4x - 1$ at the given values of x.
   - a. $x = 0$
     $f(0) = -1$
   - b. $x = -1$
     $f(-1) = 2$
   - c. $x = -3$
     $f(-3) = 20$

5. Identify the degree and leading coefficient of each of the polynomials shown.
   - a. $f(x) = 2x^5 - 8x^2 - x - 6$
     degree: 5
     leading coefficient: 2
   - b. $h(x) = x^3 - 7x^2 - x + 2$
     degree: 3
     leading coefficient: 1
   - c. $g(x) = 5x^2 - 4x^3 + 2x + 5.4$
     degree: 3
     leading coefficient: −4

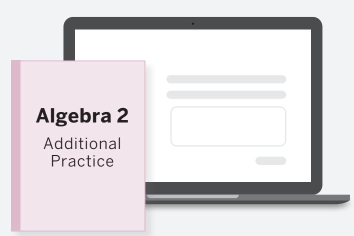
6. Multiply the polynomials shown.
   - a. $(5x)(3x + 9)$
     $15x^2 + 45x$
   - b. $(x - 1)(x + 2)$
     $x^2 + x - 2$

Practice Problem Analysis

Type	Problem	Refer to	Standard(s)	DOK
On-lesson	1	Activity 1	HSA.APR.A.1	2
	2	Activity 1	HSA.APR.A.1	2
	3	Activity 1	HSA.APR.A.1	2
Spiral	4	Unit 2, Lesson 2	HSF.IF.A.2	2
	5	Unit 2, Lesson 3	HSF.IF.A.2	2
Formative	6	Unit 2, Lesson 5	HSA.APR.A	2

Power-up: If students need additional support with the key prerequisite concept or skill this problem addresses, consider assigning the Power-up in the next lesson.

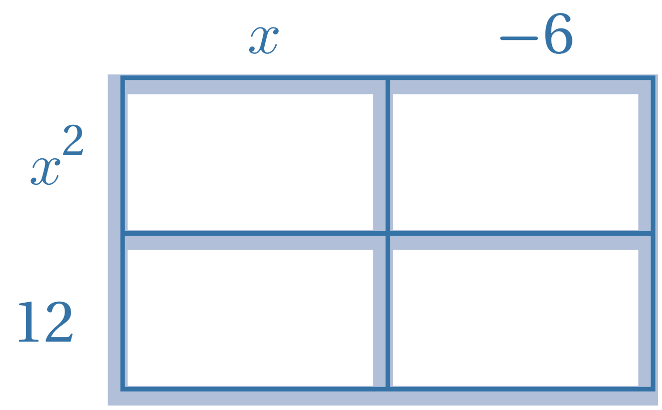
Additional Practice Available



For students who need additional practice in this lesson, assign the **Algebra 2 Additional Practice**.

Multiplying Polynomials

Let's use area diagrams and dot diagrams to multiply polynomials.



Focus

Goals

1. Identify the leading coefficient and degree of a polynomial given its standard or factored form.
2. Write the product of two or more polynomials in standard form.
3. **Language Goal:** Explain why the product of polynomials is always a polynomial. **(Reading and Writing, Speaking and Listening)**

Coherence

Today

Students build upon their skills of multiplying monomials and binomials using area diagrams. Here, they use area diagrams to multiply polynomials of higher degree and with more terms, seeing that polynomials are closed under multiplication. Students represent and multiply polynomials using dot diagrams to see similarities between integers and polynomials (**MP7**).

Previously

In Lesson 4, students practiced adding and subtracting polynomials and saw that polynomials are closed under these operations.

Coming Soon

In Lessons 7 and 8, students will connect the factors of a polynomial in factored form to the horizontal intercepts of its graph.

Rigor

- Students build **conceptual understanding** of how polynomials form a system analogous to the integers.
- Students strengthen their **fluency** in multiplying polynomials.

Standards

Addressing

HSA.APR.A.1
Understand that polynomials form a system analogous to the integers, namely, they are closed under the operations of addition, subtraction, and **multiplication**; add, subtract, and **multiply polynomials**.

Also Addressing: **HSA.APR.A**

Building On	Building Toward
HSA.APR.A	HSA.APR.B
HSA.APR.A.1	HSA.APR.B.3
HSA.SSE.A.2	

Pacing Guide

Suggested Total Lesson Time ~50 min

Warm-up	Activity 1	Activity 2	Activity 3	Summary	Exit Ticket
5 min	10 min	10 min	15 min	5 min	5 min
Independent	Independent	Pairs	Pairs	Whole Class	Independent
	MP6, MP7	MP7	MP7	MP7	
HSA.APR.A*	HSA.APR.A	HSA.APR.A, HSA.APR.A.1	HSA.APR.A, HSA.APR.A.1	HSA.APR.A, HSA.APR.A.1	HSA.APR.A, HSA.APR.A.1

In this activity, students build on their understanding of multiplying polynomials in Algebra 1

Amps powered by desmos | Activity and Presentation Slides

For a digitally interactive experience of this lesson, log in to Amplify Math at learning.amplify.com.

Practice Independent

Materials

- Exit Ticket
- Additional Practice

Math Language Development

Review words

- *coefficient*
- *degree*
- *Distributive Property*
- *factored form (of a polynomial)*
- *leading coefficient*
- *polynomial*
- *standard form (of a polynomial)*
- *terms*

Amps | Featured Activity

Activity 3

Multiplying With Dots

Students use dot diagrams to represent and multiply polynomials. Through the structure of the dot diagrams, they see how multiplying polynomials is analogous to multiplying integers.

Amps POWERED BY desmos

Building Math Identity and Community

Connecting to Mathematical Practices

Students might want to skip using the area diagram to help them multiply the polynomials. Explain to students that the area diagram model will help them make use of structure **(MP7)** and avoid errors until they become more proficient with multiplying polynomials. Remind students that using an area diagram will help them build confidence about their ability to multiply polynomials as well as give them a strategy then can return to as needed.

Modifications to Pacing

You may want to consider these additional modifications if you are short on time.

- In the **Warm-up**, Problem 3 may be omitted.
- In **Activity 2**, Problems 1 and 2 may be omitted.

Warm-up Area Diagrams

HSA.APR.A

Students activate their prior knowledge of the Distributive Property and area diagrams to multiply polynomials and write equivalent expressions.

Name: Date: Period:

Unit 2 | Lesson 5

Multiplying Polynomials

Let's use area diagrams and dot diagrams to multiply polynomials.

	x	-6
x^2		
12		

Warm-up Area Diagrams

Area diagrams can be used to visualize equivalent expressions. Study the two rectangles with side lengths of x and 4 units. Both rectangles have a width of 8 units. The area of each rectangle is shown inside.

	x	4
8	$8x$	32

1. How are the terms inside the rectangles determined?
   The terms inside each rectangle are the product of the terms that are the length and width of each rectangle.

2. Use the area diagram to determine the product of $8(x + 4)$.
   $8x + 32$

3. Complete the area diagram to represent the product of $2x(3x^2 + 4x - 8)$. Then use the diagram to write an equivalent polynomial in standard form.
   $6x^3 + 8x^2 - 16x$

	$3x^2$	$4x$	-8
$2x$	$6x^3$	$8x^2$	$-16x$

Log in to Amplify Math to complete this lesson online.

1 Launch

Activate prior knowledge by giving students 1 minute of think-time, and then discussing their response to Problem 1 with a partner prior to writing it and completing the remaining problems.

2 Monitor

Help students get started by color coding or annotating the area diagram.

Look for points of confusion:

- **Misconnecting the terms of the expressions to the dimensions of the rectangles.** Have students sketch each smaller rectangle separately and label the dimensions of each.

Look for productive strategies:

- Using the Distributive Property to check the expressions.

3 Connect

Have students share how they use an area diagram to multiply a one-term polynomial (i.e., a monomial) by a multi-term polynomial.

Highlight that the term inside a smaller rectangle represents the product of the corresponding row and column terms on the side of the diagram. The sum of the areas of the smaller rectangles is equivalent to the total area.

Ask, "How does the area diagram reflect the number of terms of the polynomials being multiplied?" The number of terms of each polynomial corresponds to the number of rows and columns of the area diagram.

Math Language Development

MLR7: Compare and Connect

After students complete Problem 1, have pairs compare their thinking and make connections to their approach to the area model. Ask:

- "What do you notice about the terms inside and outside of the rectangle?"

After students come to a consensus, instruct them to complete Problems 2 and 3.

English Learners: Annotate the area model diagram to highlight the calculations that determine the terms inside the rectangle.

Power-up

To power up students' ability to multiply polynomials, ask:

1. Here is the product of two monomials: $(2x^3)(7x^5) = 14x^8$. How do you use the coefficients and exponents to determine the product? I multiply the coefficients to determine the coefficient of the product, and add the exponents to determine the exponent of the product.
2. Use your observations to determine the following products.
   a. $(10x^4)(6x^9)$ $60x^{13}$
   b. $(3x^5)(8x^4)$ $24x^9$

Use: Before the Warm-up

Informed by: Performance on Lesson 3, Practice Problem 6 and Pre-Unit Readiness Assessment, Problems 2 and 4

MP6, MP7
HSA.APR.A

Activity 1 Products of Polynomials

Students study an area diagram to multiply polynomials and write them in standard form **(MP6)**.

Activity 1 Products of Polynomials

Area diagrams are helpful when multiplying polynomials. For each of the following products, complete an area diagram, and then use it to write the product in standard form. The first problem has been completed Show your thinking.

1. $(2x+3)(5x-1)$

	$2x$	3
$5x$	$10x^2$	$15x$
-1	$-2x$	-3

$(2x+3)(5x-1)$
$= 10x^2 - 2x + 15x - 3$
$= 10x^2 + 13x - 3$

2. $(4x-3)(x+9)$

	$4x$	-3
x	$4x^2$	$-3x$
9	$36x$	-27

$(4x-3)(x+9)$
$= 4x^2 + 36x - 3x - 27$
$= 4x^2 + 33x - 27$

3. $(x-6)(x^2+12)$

	x	-6
x^2	x^3	$-6x^2$
12	$12x$	-72

$(x-6)(x^2+12)$
$= x^3 + 12x - 6x^2 - 72$
$= x^3 - 6x^2 + 12x - 72$

4. $(x^3+10x+7)(2x-1)$

	x^3	$10x$	7
$2x$	$2x^4$	$20x^2$	$14x$
-1	$-x^3$	$-10x$	-7

$(x^3+10x+7)(2x-1)$
$= 2x^4 - x^3 + 20x^2 - 10x + 14x - 7$
$= 2x^4 - x^3 + 20x^2 + 4x - 7$

5. Now create your own area diagrams to model the product of $(x-2)(x+3)(x+5)$. Then use it to write the product in standard form. Show your thinking.

	x	-2
x	x^2	$-2x$
3	$3x$	-6

	x^2	x	-6
x	x^3	x^2	$-6x$
5	$5x^2$	$5x$	-30

$(x-2)(x+3)(x+5)$
$= (x^2 + x - 6)(x+5)$
$= x^3 + 6x^2 - x - 30$

136 Unit 2 Polynomials and Rational Functions

1 Launch

Have students examine Problem 1 and conduct the ***Think-Pair-Share*** routine to discuss how an area diagram is used to rewrite the given polynomial in standard form.

2 Monitor

Help students get started by helping them write out the sides of an area diagram and calculate a term inside the diagram.

Look for points of confusion:

- **Not realizing the terms on the diagonal are not like terms for Problem 3.** Encourage students to look carefully at each term.
- **Struggling to organize work in Problem 5**. Have students use an area diagram to multiply the first two factors. Once they determine the product, ask, "How can you use another area diagram to determine the product of *all three* factors?"

Look for productive strategies:

- Annotating the area diagram to multiply and combine terms.

3 Connect

Display Problem 5.

Have students share their strategy used to determine the product.

Ask, "What property allows you to use the area diagram to multiply?" The Distributive Property

Highlight that the area diagrams students have been using are visual representations of the Distributive Property. Each term in the first factor is multiplied by each term in the second factor. This pattern extends when multiplying three or more factors **(MP7)**.

Differentiated Support

Accessibility: *Vary Demands to Optimize Challenge*

If students need more processing time, have them focus on completing Problems 1–4 and only work on Problem 5 if they have time available.

Extension: *Math Enrichment*

Have students complete the following problem:

Write the following standard form polynomial as a product of two factors by creating an area diagram: $2x^2 + x - 6$

Sample response: $(2x-3)(x+2)$

Math Language Development

MLR5: Co-craft Questions

During the Launch, display Problem 1 without revealing any of the problems. Have students work with their partner to write 2–3 mathematical questions they could ask about the method.

Sample questions shown.

- Does this method work for all polynomials?
- What would happen if I changed the order of the length and width?
- Why are the x terms multiplied together first?

English Learners: To support students in developing metalinguistic awareness, model how to craft a mathematical question. Consider displaying one of the sample questions.

Pairs | 10 min

Activity 2 The Leading Coefficient and Degree of Products

MP7
HSA.APR.A, HSA.APR.A.1

Students multiply polynomials to identify how the leading coefficient and degree of the product relate to those of the factors, and to discuss polynomials' closure under multiplication.

Name: ______ Date: ______ Period: ______

Activity 2 The Leading Coefficient and Degree of Products

Use the polynomials shown as you complete the problems.

Polynomial A	Polynomial B	Polynomial C
$2x^4 + 3x$	$5x^2 - 7x$	$-4x^2 - x$

1. Determine the product of Polynomials A and B. Write the product in standard form.
   $10x^6 - 14x^5 + 15x^3 - 21x^2$

2. What is the leading coefficient and degree of the product of Polynomials A and B?
   The leading coefficient is 10. The degree is 6.

3. Determine the product of Polynomials A, B, and C. Write the product in standard form.
   $-40x^8 + 56x^7 - 60x^5 + 84x^4 - 10x^7 + 14x^6 - 15x^4 + 21x^3$
   $= -40x^8 + 46x^7 + 14x^6 - 60x^5 + 69x^4 + 21x^3$

4. What is the leading coefficient and degree of the product of Polynomials A, B, and C?
   The leading coefficient is −40. The degree is 8.

5. How might you determine the leading coefficient and degree of the product of polynomials without actually calculating the product?
   Sample response: First, place each polynomial being multiplied in standard form. The leading coefficient of the product equals the product of the leading coefficients. The degree of the product equals the sum of the degrees.

6. In Lesson 4, you saw that adding and subtracting polynomials always results in a polynomial. Do you think the product of two polynomials is always a polynomial? Explain your thinking.
   Yes. Sample response: The terms of the product of polynomials will still be a sum of multiples of powers of x.

1 Launch

Have students complete Problems 1 and 2 independently, and have students share their responses. Review the definition of *leading coefficient* and *degree* before students independently complete the remainder of the activity.

2 Monitor

Help students get started by having them construct an area diagram and complete the sides of the diagram.

Look for points of confusion:

- **Always using the first coefficient in the polynomial as the leading coefficient.** Have students identify the term with the greatest exponent. Also, suggest students write polynomials in standard form.

Look for productive strategies:

- Checking their answer in Problem 5 by multiplying polynomials given in the activity.

3 Connect

Display Polynomials A, B, and C.

Ask, "What is the least information we need about the factors to determine the leading coefficient and degree of the polynomial?" We need only the leading coefficient and degree of each factor.

Have individual students share their responses to Problems 4 and 5.

Highlight that the product of two or more polynomials is always a polynomial. Terms with the same variable and exponent are combined, and the result is a term that is still a multiple of a power of x **(MP7)**.

Differentiated Support

Accessibility: *Vary Demands to Optimize Challenge*

After students have independently completed Problems 1 and 2, use a think-aloud to model and demonstrate to students how you would determine the product of Polynomials A and B and the leading coefficient of the product. Ask students to compare their approach to your approach.

Extension: *Math Enrichment*

Have students complete the following problem:

Examine the leading coefficient and degree of the following monomial. Create 3 polynomials that, when multiplied, will have the same leading coefficient and degree of the monomial.

Monomial: $-36x^7$

Sample response: Polynomial A: $(-2x^3 + x^2)$
Polynomial B: $(3x^2 + x - 1)$
Polynomial C: $(6x^2 + 5)$

Math Language Development

MLR1: Stronger and Clearer Each Time

After students complete Problem 5, have pairs meet with 1–2 other pairs of students to share their responses. Encourage reviewers to ask clarifying questions, such as:

- "How did you determine the leading coefficient in Problem 3?"
- "Why did you put the polynomials in standard form?"
- "Does this method always work? How do you know?"

Have students revise their responses, as needed.

English Learners: Use intentional grouping so that students with different English language proficiency levels can interact and have an opportunity to listen to peers with more advanced proficiency.

Pairs | 15 min

Activity 3 Multiplying With Dots

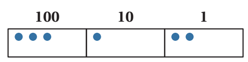
MP7
HSA.APR.A, HSA.APR.A.1

Students use dot diagrams to model multiplication of polynomials and see how they are analogous to integers written in base 10 form **(MP7).**

Amps Featured Activity Multiplying With Dots

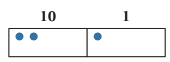
Activity 3 Multiplying With Dots

Recall how to represent numbers with a visual dot diagram. 312 has 3 dots in the 100s place, 1 dot in the 10s place, and 2 dots in the 1s place.



When multiplying numbers, such as $312 \cdot 21$, replace each dot in the representation of 312 with a group of dots that represents 21.

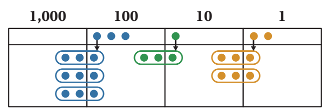
Start from the right, and replace every dot with the groups of dots that represent 21, or



- The 2 dots in the 1s place are replaced with 2 groups of 21 dots.
- The 1 dot in the 10s place is replaced with 1 group of 21 dots.
- The 3 dots in the 100s place are replaced with 3 groups of 21 dots.

1. How does the dot diagram shown reveal that the product of $312 \cdot 21$ is 6,552?

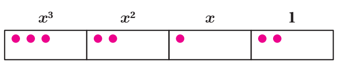
   Sample response: The total number of dots in each column represents the digit in the place value that the column represents.



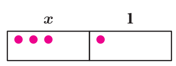
Polynomial multiplication can also be represented by dot diagrams. Consider the product of $(3x^3 + 2x^2 + x + 2)(3x + 1)$.

2. Represent each factor with a dot diagram.

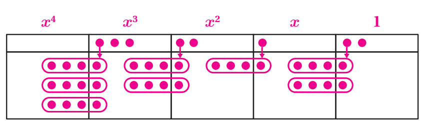
   $(3x^3 + 2x^2 + x + 2)$



   $(3x + 1)$



3. Now replace every dot in your representation of $(3x^3 + 2x^2 + x + 2)$ with the dots in your representation of $(3x + 1)$.



4. Use your diagram to write the product of $(3x^3 + 2x^2 + x + 2)(3x + 1)$.

   $9x^4 + 9x^3 + 5x^2 + 7x + 2$

STOP

138 Unit 2 Polynomials and Rational Functions

1 Launch

Review the beginning prompt and the dot diagram for 312 together. Activate prior knowledge by asking, "How do we represent the polynomial $3x^2 + x + 2$ using a dot diagram?"

2 Monitor

Help students get started by helping them construct a dot diagram for each polynomial.

Look for points of confusion:

- **Replacing each dot of the four-term polynomial with the dots from the binomial in the incorrect column.** To organize the dot diagram, it might be helpful to circle or highlight each group of dots from the binomial as the dots are replaced in the four-term polynomial with the dots.

Look for productive strategies:

- Aligning groups of dots in the product's diagram.
- Replacing each dot of the binomial with the dots from the four-term polynomial.

3 Connect

Display the representation of the product of the two polynomials.

Have individual students share their approach to using dot diagrams for multiplication.

Highlight that replacing each dot in the four-term polynomial with the dots from the binomial represent multiplying each term by both terms from the binomial.

Ask:

- "What would a dot diagram that models the product of a polynomial by a trinomial look like?" Each dot would be replaced with dots that span over three columns in the dot diagram.
- "How does multiplying using dot diagrams show that polynomials are similar to integers?"

Differentiated Support

Accessibility: *Optimize Access to Technology*

Have students use the Amps slides for this activity, in which they can use dot diagrams to represent and multiply polynomials, in order to see how multiplying polynomials is analogous to multiplying integers.

Accessibility: *Activate Prior Knowledge*

During the Launch, remind students of the work they completed in Lesson 2 when they represented polynomials with dot diagrams. Make available sample student work from Lesson 2 to reinforce the work they have already done, in order to prepare them for multiplying with dot diagrams in this lesson.

Whole Class | 5 min

MP7
HSA.APR.A, HSA.APR.A.1

Summary

Review and synthesize multiplying polynomials, determining the lead coefficient and degree of polynomials written in factored form, and how polynomials are closed under multiplication.

Name: ______ Date: ______ Period: ______

Summary

In today's lesson . . .

You used area diagrams to multiply polynomials and write their product in standard form.

For example, to expand the factored polynomial $p(x) = (2x^3 - 5x)(-4x^2 + 6)$, you can use an area diagram to determine that it equals: $p(x) = -8x^5 + 12x^3 + 20x^3 - 30x$, or $p(x) = -8x^5 + 32x^3 - 30x$.

	$2x^3$	$-5x$
$-4x^2$	$-8x^5$	$20x^3$
$+6$	$12x^3$	$-30x$

You also discovered a way to efficiently determine the degree and leading coefficient of the product of polynomials. The degree of the product is the sum of the degrees of the factors, while the leading coefficient of the product is the product of the leading coefficients of the factors.

For example, with $p(x) = (2x^3 - 5x)(-4x^2 + 6)$, the leading coefficient of p is -8 because $2 \cdot (-4) = -8$. The degree of p is 5 because $3 + 2 = 5$.

Reflect:

Synthesize

Display an area diagram representing the polynomial $p(x) = (2x^3 - 5x)(-4x^2 + 6)$.

Have students share how to use area diagrams to multiply polynomials and how to determine the leading coefficient and degree without multiplying the factors **(MP7)**.

Highlight that when multiplying two integers, the result is always an integer. The same is true for polynomials. Multiplying polynomials will always result in a polynomial.

Ask, "Why might it be helpful to know the leading coefficient and degree of a polynomial?" Sample response: The leading coefficient and degree reveal information about the shape of the ends of the graph.

Reflect

After synthesizing the concepts of the lesson, allow students a few moments for reflection. Encourage them to record any notes in the *Reflect* space provided in the Student Edition. To help them engage in meaningful reflection, consider asking:

- "How do area diagrams represent the Distributive Property?"

Independent | 5 min

Exit Ticket

HSA.APR.A, HSA.APR.A.1

Students demonstrate their understanding by expanding a polynomial from factored form to standard form, and identifying its leading coefficient and degree.

Printable

Name: Date: Period:

Exit Ticket 2.05

Consider the polynomial function $P(x) = (2x + 6)(3x + 3)(x - 1)$.

1. Write the function in standard form.
   $P(x) = 6x^3 + 18x^2 - 6x - 18$

2. Identify the leading coefficient and degree of the function.
   The leading coefficient is 6. The degree is 3.

Self-Assess

1 I don't really get it | 2 I'm starting to get it | 3 I got it

a. I can write the product of two or more polynomials in standard form. 1 2 3

b. I can identify the leading coefficient and degree of polynomials, given their standard or factored forms. 1 2 3

c. I understand that multiplying polynomials always results in another polynomial. 1 2 3

Unit 2 Lesson 5 Multiplying Polynomials

Success looks like . . .

» **Goal:** Identifying the leading coefficient and degree of a polynomial given its standard or factored form.

✓ Identifying the leading coefficient and degree of $P(x)$ in Problem 2.

» **Goal:** Writing the product of two or more polynomials in standard form.

✓ Correctly expanding $P(x)$ in Problem 1.

» **Language Goal:** Explaining why the product of polynomials is always a polynomial. **(Reading and Writing, Speaking and Listening)**

Suggested next steps

If students inaccurately determine the product or do not write the product in standard form in Problem 1, consider:

- Reviewing using area diagrams to multiply polynomials from Activity 1.
- Assigning Practice Problem 1.
- Asking, "How could you use two different area diagrams to help multiply the polynomial expressions?"

If students inaccurately identify the leading coefficient and degree in Problem 2, consider:

- Reviewing how to use the leading coefficient and degree of each term to determine the leading coefficient and degree of the product from Activity 2.
- Assigning Practice Problems 1 and 3.
- Asking, "How might you be able to determine the leading coefficient and degree without writing the polynomial in standard form?"

Professional Learning

This professional learning moment is designed to be completed independently or collaboratively with your fellow mathematics educators. Prompts are provided so that you can reflect on this lesson before moving on to the next lesson.

Points to Ponder . . .

- What worked and didn't work today? In this lesson, students used dot diagrams to multiply polynomials. How will that support students seeing polynomials analogous to the integer set?
- In this lesson, students multiplied polynomials. How will that support dividing polynomials in later lessons? What might you change the next time you teach this lesson?

Math Language Development

MLR1: Stronger and Clearer Each Time

In Activity 2, you used intentional grouping with MLR1 to group students with different English language proficiency levels. Use these prompts to reflect on this routine.

- What effect did this grouping strategy have on student revisions?
- Would you change anything the next time you use MLR1?

Practice

Independent

Name: ______ Date: ______ Period: ______

Practice

1. Write each polynomial function in standard form. Then state its degree and leading coefficient.

a $h(x) = (4x^3 - 2x)(-7x^2 - 10)$

$h(x) = -28x^5 - 26x^3 + 20x$

The degree is 5. The leading coefficient is -28.

b $f(x) = (x + 1)(x + 3)(x - 4)$

$f(x) = x^3 - 13x - 12$

The degree is 3. The leading coefficient is 1.

c $g(x) = (3x + 1)(x + 3)(x - 4)$

$g(x) = 3x^3 - 2x^2 - 37x - 12$

The degree is 3. The leading coefficient is 3.

2. Elena writes the function $p(x) = 4(2x^2 - 3) - 8x^2$ in standard form, and then claims that the function is not a polynomial because it is a constant. Do you agree? Explain your thinking.

I disagree with Elena. Sample response: The function written in standard form is $p(x) = -12$. Elena is correct that the function is a constant, but constants are polynomials. $p(x)$ is a polynomial with a degree of 0.

3. Tyler examined the polynomial $(-3x^2)(2x + 5)(5x^3 - 6)(-2x^5 - 4x^4 - 5x + 90)$ and determined the leading coefficient is 2 and the degree is 30. Do you agree? Explain your thinking.

I disagree with Tyler. Sample response: The leading coefficient is determined by multiplying the leading coefficients of the factors. The leading coefficient is 60 because $(-3) \cdot 2 \cdot 5\,(-2) = 60$. The degree is determined by adding the degrees of the factors. The degree is 11 because $2 + 1 + 3 + 5 = 11$.

Name: ______ Date: ______ Period: ______

Practice

4. Write the following polynomial functions in standard form.

a $f(x) = -3x^2 + 4x^4 - 3x + 10 - 11x^2$

$f(x) = 4x^4 - 14x^2 - 3x + 10$

b $g(x) = -x^5 + x^4 - 3x^3 - 10x^5 + 61x^4$

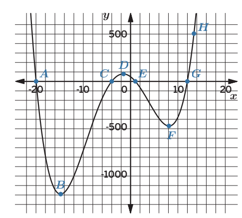
$g(x) = -11x^5 + 62x^4 - 3x^3$

c $h(x) = (6x^3 - 5x) - (-6x - 5x^3)$

$h(x) = 11x^3 + x$

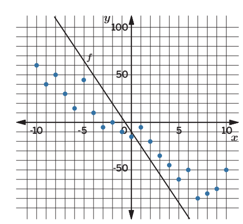
5. The graph of the function f is shown. Identify the points that are the local maxima or local minima of f.

The local maximum is point D. The local minima are points B and F.



6. Kiran created a scatter plot of data from a science experiment. He then graphed the function f, which he says is the line of best fit for the data. Do you agree that f is the line of best fit for the data? Explain your thinking.

I do not agree that f is the line of best fit. Sample response: The line of best fit minimizes the sum of the squared distances between the points and the line. A line with the same horizontal intercept, but a shallower slope would better fit the data.

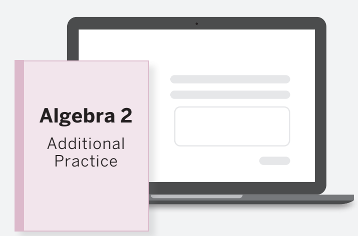


Practice Problem Analysis

Type	Problem	Refer to	Standard(s)	DOK
On-lesson	1	Activities 1 and 2	HSA.APR.A	2
	2	Activity 2	HSA.APR.A, HSA.APR.A.1	2
	3	Activity 2	HSA.APR.A, HSA.APR.A.1	2
Spiral	4	Unit 2 Lesson 4	HSA.APR.A.1	2
	5	Unit 2 Lesson 3	HSF.IF.B.4	2
Formative	6	Unit 2 Lesson 6	HSS.ID.B.6.C	2

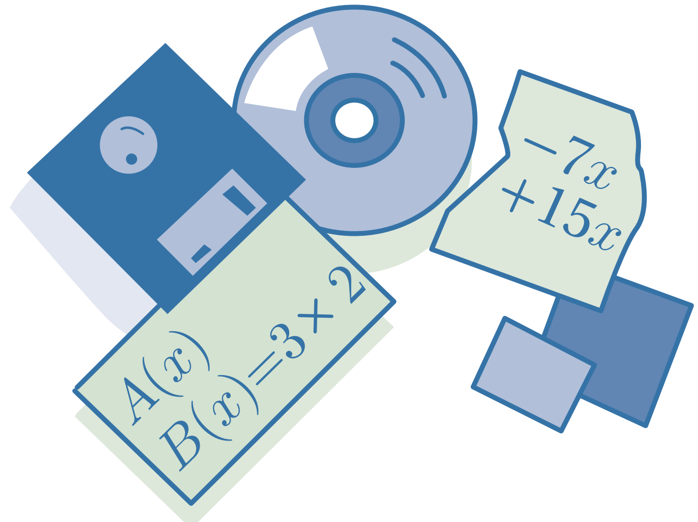
Power-up: If students need additional support with the key prerequisite concept or skill this problem addresses, consider assigning the Power-up in the next lesson.

Additional Practice Available



For students who need additional practice in this lesson, assign the **Algebra 2 Additional Practice**.

The Remainder Theorem



Let's explore what happens when a polynomial is divided by one of its factors.

Focus

Goals

1. Comprehend that, for a polynomial $p(x)$ and a number a, the remainder on division by $x - a$ is $p(a)$.
2. Comprehend that, for a polynomial $p(x)$, $x - a$ is a factor if $p(a) = 0$ and, conversely, that $p(a) = 0$, if $x - a$ is a factor.

Rigor

- Students develop **conceptual understanding** of the Remainder Theorem.
- Students **apply** their understanding of the remainder theorem to determine all the factors of a polynomial.

Coherence

Today

Students divide polynomials by other polynomials and conclude that the divisor is not a factor of the original polynomial when there is a remainder. They make connections between the remainder of a division problem and the function's value while learning about the *Remainder Theorem*, which states that $p(x)$ divided by $(x - a)$ results in a remainder of $p(a)$. They use the Remainder Theorem to evaluate polynomials and determine that when $p(a) = 0$ for a specific polynomial p, $(x - a)$ is a factor of the polynomial.

Previously

In previous lessons, students used diagrams and polynomial long division to divide polynomials by known factors to determine unknown factors.

Coming Soon

In the next several lessons, students will investigate rational functions, asymptotes, and the end behavior of rational functions, using strategies they developed while working with polynomials.

Standards

Addressing

HSA.APR.B.2
Know and apply the Remainder Theorem: For a polynomial $p(x)$ and a number a, the remainder on division by $(x - a)$ is $p(a)$, so $p(a) = 0$ if and only if $(x - a)$ is a factor of $p(x)$.

Also addressing: **HSA.APR.A.1**

Building On	Building Toward
HSA.APR.A.1	HSA.APR.B.2

Pacing Guide

Suggested Total Lesson Time **~50 min**

Warm-up	Activity 1	Activity 2	Summary	Exit Ticket
5 min	20 min	15 min	5 min	5 min
Independent	Pairs	Pairs	Whole Class	Independent
MP8	MP7, MP8	MP1		
HSA.APR.A.1	HSA.APR.B.2, HSA.APR.A.1	HSA.APR.B.2	HSA.APR.B.2	HSA.APR.B.2

Amps powered by desmos | Activity and Presentation Slides

For a digitally interactive experience of this lesson, log in to Amplify Math at **learning.amplify.com**.

Practice Independent

Materials

- Exit Ticket
- Additional Practice
- Anchor Chart PDF, *Remainder Theorem*

Math Language Development

New words

- ***Remainder Theorem***

Review words

- *dividend*
- *divisor*
- *factor*
- *quotient*
- *remainder*

Amps | Featured Activity

Activity 1

Interactive Dot Diagram

Students explore the Remainder Theorem with a digital interactive dot diagram and make connections between the *dividend, divisor, quotient,* and *remainder.*

Amps POWERED BY desmos

Building Math Identity and Community

Connecting to Mathematical Practices

Students may struggle with finding an entry point for determining the value of u. Explain to students that it is important to make sense of the problem and persevere in solving it **(MP1)**. Encourage them to pause and assess what they know rather than simply jumping into a solution attempt. Suggest they start with what it means for a polynomial to have a given factor and think about what that means for a remainder.

Modifications to Pacing

You may want to consider this additional modification if you are short on time.

- In **Activity 1**, omit polynomial function $E(x)$.

Independent | 5 min

Warm-up Notice and Wonder

MP8
HSA.APR.A.1

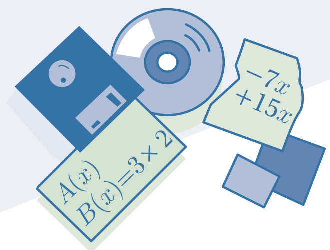
Students compare diagrams depicting long division with integers and polynomials to prepare them to consider cases in which there are remainders **(MP8).**

Name: ______ Date: ______ Period: ______

Unit 2 | Lesson 15

The Remainder Theorem

Let's explore what happens when a polynomial is divided by one of its factors.



Warm-up Notice and Wonder

Consider the following division problems. What do you notice? What do you wonder?

$$\begin{array}{r} 47 \\ 7\overline{)330} \\ -\,28 \\ \hline 50 \\ -\,49 \\ \hline 1 \end{array} \qquad \begin{array}{r} 33 \\ 10\overline{)330} \\ -\,30 \\ \hline 30 \\ -\,30 \\ \hline 0 \end{array} \qquad \begin{array}{r} 3x^2 + 4x - 7 \\ x - 2\overline{)3x^3 - 2x^2 - 15x + 14} \\ -\,(3x^3 - 6x^2) \\ \hline 4x^2 - 15x \\ -\,(4x^2 - 8x) \\ \hline -\,7x + 14 \\ -(-7x + 14) \\ \hline 0 \end{array} \qquad \begin{array}{r} 3x^2 - 5x + 10 \\ x + 1\overline{)3x^3 - 2x^2 - 15x + 14} \\ -\,(3x^3 + 3x^2) \\ \hline -5x^2 - 15x \\ -\,(-5x^2 - 5x) \\ \hline -\,10x + 14 \\ -(10x + 10) \\ \hline 4 \end{array}$$

1. I notice . . .
   Sample responses:
   - 7 does not divide evenly into 330 because there is a remainder 1.
   - 10 is a factor of 330 because there is a remainder of 0.
   - $x - 2$ is a factor of the polynomial $3x^3 - 2x^2 - 15x + 14$.
   - It seems like $x + 1$ does not divide evenly into the polynomial $3x^3 - 2x^2 - 15x + 14$ because there is a remainder of 4.

2. I wonder . . .
   Sample responses:
   - Why is there a 4 remaining in the last problem?
   - Can you completely factor the polynomial $3x^3 - 2x^2 - 15x + 14$?

Log in to Amplify Math to complete this lesson online.

1 Launch

Display the four division problems. Conduct the *Notice and Wonder* routine and remind students there are no incorrect answers.

2 Monitor

Help students get started by asking, "How are the diagrams similar and how are they different?"

Look for points of confusion:

- **Misunderstanding how the remainder relates to the dividend and divisor.** Ask students to find the product of 7 and 47 in Problem A and relate this to 330 and 1. $7 \cdot 47 = 329$, so $330 = 7 \cdot 47 + 1$.

Look for productive strategies:

- Recognizing that the quotients in Problems B and C have no remainder, so the divisor is a factor of the dividend.

3 Connect

Display the diagrams in Problems C and D and write dividend = divisor • quotient + remainder.

Define *divisor, dividend, quotient* and *remainder* and label the corresponding values in the polynomial diagrams.

Highlight how multiplication and division with polynomials is analogous to the same integer operations.

Ask, "How can you show that the dividend in Problem D is equivalent to the expression $(x + 1) \cdot (3x^2 - 5x + 10) + 4$?" Sample response: Multiply the divisor and quotient, and then add the remainder:
$(x + 1)(3x^2 - 5x + 10) + 4$
$= (3x^2 - 2x^2 + 5x + 10) + 4$
$= 3x^3 - 2x^2 + 5x + 14$

Math Language Development

MLR2: Collect and Display

During the Launch, collect student language they use to describe what they notice and wonder about the division problems. Press students for more details by asking, "How do you know $(x + 1)$ is not a factor of the polynomial in the last problem?" Add an example of the polynomial long division to the class display and annotate it with student language. Use this time to define the terms *divisor, dividend, quotient,* and *remainder.*

English Learners: Give students time to rehearse and formulate a response before sharing with the whole class.

Power-up

To power up students' ability to rewrite integer division as multiplication, ask:

What is $330 \div 4$? Express your answer using multiplication and addition.
$330 = 4 \cdot 82 + 2$

Use: Before the Warm-up

Informed by: Performance on Lesson 14, Practice Problem 6

Pairs | 20 min

Activity 1 Is There Anything Leftover?

MP7, MP8
HSA.APR.B.2

Students complete repeated calculations to make the connection that the value of polynomial p when $x = a$, or $p(a)$, is equal to the value of the remainder when $p(x)$ is divided by $x - a$ **(MP8).**

Amps Featured Activity Interactive Dot Diagram

Activity 1 Is There Anything Left Over?

When sending or storing digital information, there is always a risk of losing some data. Error correcting codes have been in use since the 1960s to prevent such things as blurry images or bad reception. Polynomial division algorithms are an essential part of these codes, and are still being used today by mathematicians such as Christina Eubanks-Turner. Let's see how polynomial division and remainders can provide important information about a polynomial.

1. Evaluate the following polynomial functions at $x = 2$.

$A(x) = 6x^2 - 7x - 5$	$A(2) = 5$
$B(x) = 3x^2 + 15x - 42$	$B(2) = 0$
$C(x) = 3x^3 - 2x^2 - 15x + 14$	$C(2) = 0$
$D(x) = 2x^3 + 13x^2 + 16x + 5$	$D(2) = 105$
$E(x) = x^4 + 5x^3 - 27x^2 - 101x - 70$	$E(2) = -324$

2. Which of the polynomials from Problem 1 do you think could have $x - 2$ as a factor? Explain your thinking.

$B(x)$ and $C(x)$. Sample response: $x = 2$ is a zero of these functions.

3. Divide each polynomial from Problem 1 by $x - 2$. State the quotient for each division problem and give any remainder.

a $(6x^2 - 7x - 5) \div (x - 2)$

$= 6x + 5$, remainder 5

b $(3x^2 + 15x - 42) \div (x - 2)$

$= 3x + 21$

c $(3x^3 - 2x^2 - 15x + 14) \div (x - 2)$

$= 3x^2 + 4x - 7$

d $(2x^3 + 13x^2 + 16x + 5) \div (x - 2)$

$= 2x^2 + 17x + 50$, remainder 105

e $(x^4 + 5x^3 - 27x^2 - 101x - 70) \div (x - 2)$

$= x^3 + 7x^2 - 13x - 127$, remainder -324

1 Launch

Read the narrative as a class and have students work independently on Problems 1 and 2. Select a student to explain their reasoning for which polynomials have $x - 2$ as a factor.

2 Monitor

Help students get started by activating prior knowledge by asking, "How do you know when a divisor is a factor of a polynomial?"

Look for points of confusion:

- **Incorrectly reasoning about factors in Problem 2.** Ask students to check their answers by using long division to analyze the remainder or by graphing the polynomials to identify the x-intercepts.

Look for productive strategies:

- Using repeated reasoning to conclude that the remainder of each polynomial divided by $x - a$ is equal to the polynomial evaluated at $p(a)$.

Activity 1 continued >

Differentiated Support

Accessibility: *Optimize Access to Technology*

Have students use the Amps slides for this activity, in which they can explore the Remainder Theorem with a digital interactive dot diagram and make connections between the *dividend, divisor, quotient,* and *remainder.*

Accessibility: *Vary Demands to Optimize Challenge*

Chunk this task into smaller, more manageable parts by differentiating the degree of difficulty or complexity. For example, invite students to choose and respond to 3 of the 5 polynomials in Problem 3.

Math Language Development

MLR2: Collect and Display

Display the question from the "Ask." Conduct a ***Think-Pair-Share*** routine to give partners time to respond to the question. As students share their response, listeners should press for details by asking, "Can you give an example?" or "How do you know?" Circulate as partners share their responses and collect student language that you hear. Add this language to the class display, along with the term *Remainder Theorem* and its definition.

English Learners: Add an example of polynomial long division and a dot diagram to the class display to show how they are similar and different. This will help students further internalize the Remainder Theorem.

Pairs | 20 min

Activity 1 Is There Anything Leftover? (continued)

MP7, MP8
HSA.APR.B.2, HSA.APR.A.1

Students complete repeated calculations to make the connection that the value of polynomial p when $x = a$, or $p(a)$, is equal to the value of the remainder when $p(x)$ is divided by $x - a$ **(MP8).**

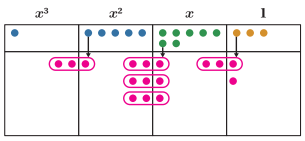
Name: Date: Period:

Activity 1 Is There Anything Left Over? (continued)

4. How do the remainders from each polynomial division problem compare to the values of each function at $x = 2$?

Sample response: When dividing by $x - 2$, the remainder is the same as the value of the polynomial at $x = 2$.

5. Use a dot diagram to show that the remainder of this division problem $(x^3 + 5x^2 + 7x + 3) \div (x + 2) = ?$ is 1.



a State the quotient of the division problem and give any remainder.

$x^2 + 3x + 1$, remainder 1

b Where do you see the remainder in the diagram?

There is one dot in the last column that is not circled.

c Rewrite your response for part a using the divisor, quotient, and remainder:

$x^3 + 5x^2 + 7x + 3 = (x + 2) \cdot (x^2 + 3x + 1) + 1$

6. Let $f(x) = x^3 + 5x^2 + 7x + 3$.

a Evaluate $f(-2)$. How does this compare to your response for Problem 5a?
$f(-2) = 1$. Sample response: When I divided f by $x + 2$, the remainder is the same as the value of $f(-2)$.

b Evaluate $f(-3)$. What information does this value tells you about the polynomial?
$f(-3) = 0$. Sample response: The value of $f(-3)$ tells me that when f is divided by $x + 3$ the remainder is 0, so $x + 3$ must be a factor of f.

Featured Mathematician

Christina Eubanks-Turner

Christine Eubanks-Turner is a mathematics professor at Loyola Marymount University. Her research on using polynomial division algorithms to decode digital information has a wide range of applications in data storage and communications, such as bar codes, optical discs, cell phones, high-speed modems, and satellite communications.

3 Connect

Display Problem 4.

Ask, "When $p(x)$ is divided by $x - a$ why does it seem like the remainder is equal to $p(a)$?"

Have individual students share their responses. Select and sequence students who can explain why this seems true.

Highlight the relationship between division and multiplication where the dividend = divisor • quotient + remainder, so $p(x) = (x - a) \cdot q(x) + r$ **(MP7)**.

Ask, "What is $p(x)$ when $x = a$?"
$p(a) = (a - a) \cdot q(a) + r$ or $p(a) = r$

Define the ***Remainder Theorem*** for polynomials which says that, for a polynomial $p(x)$ and a number a, the remainder on division by $x - a$ is $p(a)$. Display the Anchor Chart PDF, *Remainder Theorem*.

Featured Mathematician

Christina Eubanks-Turner

Christina Eubanks-Turner is a mathematics professor who works to improve the efficiency of computer codes to store and read digital data, which is based on polynomial long division algorithms.

Pairs | 15 min

MP1
HSA.APR.B.2

Activity 2 The Unknown Coefficient

Students reinforce their understanding of the Remainder Theorem by solving for a missing variable in the equation of a function **(MP1).**

Activity 2 The Unknown Coefficient

Consider the polynomial function $f(x) = x^4 - ux^3 + 24x^2 - 32x + 16$, where u is an unknown real number. If $x - 2$ is a factor of f, what is the value of u? Explain how you know.
$u = 8$; Sample response: Since $x - 2$ is a factor, $x = 2$ must be a zero of the function and so $f(2) = 0$. I can substitute $(2, 0)$ into the function's equation and then solve for u.

$0 = (2)^4 - u(2)^3 + 24(2)^2 - 32 \cdot 2 + 16$

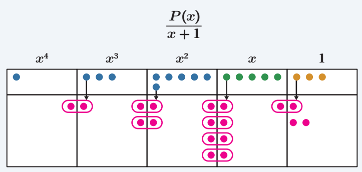
$0 = 16 - 8u + 96 - 64 + 16$

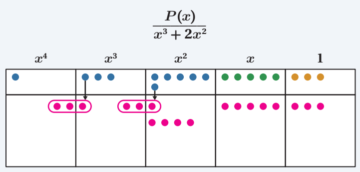
$0 = 64 - 8u$

$u = 8$

Are you ready for more?

Here are some dot diagrams that show the 4th degree polynomial, $P(x) = x^4 + 3x^3 + 6x^2 + 5x + 3$, divided by a linear factor and by a cubic factor.





1. Complete the diagrams to determine the remainder for each division problem.
   a. $\frac{P(x)}{x+1} = ?$ $\frac{P(x)}{x+1} = x^3 + 2x^2 + 4x + 1$, remainder 2
   b. $\frac{P(x)}{x^3+2x^2} = ?$ $\frac{P(x)}{x^3+2x^2} = x + 1$, remainder $4x^2 + 5x + 3$
2. For each division, how does the degree of the remainder compare to the degree of the divisor? The degree of the remainder is one less than that of the divisor for Problems 1a and 1b.
3. Could the remainder ever have the same degree as the divisor, or a higher degree? Give an example to explain your thinking. No. Sample response: For example, if the dividend has a degree of 9 and the divisor has a degree of 4, then the quotient would have a degree of 5 and the remainder would have a degree less than 4 (the degree of the divisor).
   $\frac{x^9 + x^8 - x^7}{x^4 + x^3} = x^5 - x^3 + x^2 - x + 1$, remainder $-x^3$

STOP

1 Launch

Read the problem together as a class. Conduct the *Think-Pair-Share* routine.

2 Monitor

Help students get started by asking, "What must be true about $f(2)$ if $x - 2$ is a factor of $f(x)$? $f(2) = 0$

Look for points of confusion:

- **Misunderstanding how the divisor and the remainder are related when $x - 2$ is a factor.** Ask students to summarize the Remainder Theorem in their own words when the remainder is zero.

Look for productive strategies:

- Graphing the function with different values of u and checking to see if $x = 2$ is a zero of the polynomial.
- Using long division with different values of u and checking to see if the remainder is 0.
- Substituting $x = 2$, $f(2) = 0$ and solving for u.

3 Connect

Have student pairs share responses to the problem. Select and sequence students who used different methods to determine u and record their strategies.

Ask, "How do you know that $f(2) = 0$ when $x - 2$ is a factor of the polynomial? How can you prove it?"

Highlight how to use the Remainder Theorem to analyze $f(x)$ divided by $x - 2$.
$f(x) = (x - 2) \cdot q(x) + r$, so if $x - 2$ is a factor then $r = 0$. When $x = 2$, show that the result is:
$f(2) = (2 - 2) \cdot q(a) + 0$ or $f(2) = 0$. Conclude that if $f(a) = 0$, then $x - a$ is a factor of $f(x)$.

Differentiated Support

Accessibility: *Bridge Knowledge Gaps*

Differentiate the degree of difficulty by beginning with an example with more accessible values. Highlight connections between representations by starting with a simpler problem, such as $x^2 + ux + 10$, with a known factor of $(x + 5)$ where $u = 7$.

Math Language Development

MLR7: Compare and Connect

Use this routine to prepare students for the whole-class discussion. At the appropriate time, ask students to prepare a visual display that shows their mathematical thinking and reasoning for the given question. Invite students to quietly circulate and read at least 2 other posters or visual displays in the room. Give students 2 minutes of quiet think-time to consider what is the same and what is different about the displays. Next, ask students to find a partner to discuss what they noticed. Listen for and amplify observations involving the strategy of substituting 2 for x. This will help students connect their understanding of *linear factors, zeros of a function,* and *function notation* as they discuss different strategies.

Whole Class | 5 min

Summary

HSA.APR.B.2

Review and synthesize how the Remainder Theorem is used to determine the zeros of a polynomial function.

Name: ______ Date: ______ Period: ______

Summary

In today's lesson . . .

You divided one polynomial by another and looked at whether or not the divisor was a factor. During this investigation, you discovered that if the polynomial function $P(x)$ is divided by $x - a$, then the remainder is $P(a)$. The ***Remainder Theorem*** for polynomials states that $\frac{P(x)}{x-a}$ always has a remainder of $P(a)$. Therefore, if division by $(x - a)$ results in a remainder of 0, then $P(a) = 0$.

It is more efficient to check whether a particular value of x is a zero of a polynomial function. Applying the *Remainder Theorem*, substitute $x = a$ into $P(x)$. If $P(a) = 0$, then $x = a$ is a zero of P, which means $x - a$ is a factor of the polynomial.

For example, consider the polynomial function $P(x) = x^3 + 5x^2 + 7x + 3$. Is $x = 2$ a zero of P?

If you evaluate $P(x)$ at $x = 2$, the result is:

$$\begin{aligned} P(2) &= (2)^3 + 5(2)^2 + 7(2) + 3 \\ &= 8 + 20 + 14 + 3 \\ &= 45 \end{aligned}$$

Because the remainder is non-zero, $x - 2$ is not a factor of $P(x)$. According to the *Remainder Theorem*, the quotient has a remainder of 45.

$P(x) = (x - 2)(x^2 + 7x + 21) + 45$.

Another way to express this polynomial division would be to write it as $\frac{P(x)}{b(x)} = q(x) + \frac{r}{b(x)}$, where $b(x)$ is the divisor, $q(x)$ is the quotient, and r is the remainder.

For this example, $\frac{P(x)}{(x-2)} = (x^2 + 7x + 21) + \frac{45}{x-2}$.

Reflect:

Synthesize

Display the polynomials from parts c and d of the Warm-up.

Have students share how they would apply the Remainder Theorem to determine all the zeros of the polynomial $3x^3 - 2x^2 - 15x + 14$.

Highlight that the Remainder Theorem is used to more efficiently determine factors of a polynomial by testing which values of x are zeros of the polynomial. Using a known factor and polynomial long division, you can rewrite the polynomial as a product of linear factors.

Ask, "Does polynomial long division that results in a remainder give you any useful information?" Sample response: If division by $x - a$ results in a remainder then you can eliminate $x = a$ as a possible zero of the polynomial. In addition, you know that $P(a)$ is equal to the remainder so you also know a point on the graph of the function where a is the x-coordinate and the remainder is the y-coordinate.

Formalize vocabulary: *Remainder Theorem*

Reflect

After synthesizing the concepts of the lesson, allow students a few moments for reflection. Encourage them to record any notes in the *Reflect* space provided in the Student Edition. To help them engage in meaningful reflection, consider asking:

- "How would you explain to a friend, who is absent today, what the Remainder Theorem is and why it is a useful tool for analyzing polynomials?"

Math Language Development

MLR2: Collect and Display

As students formalize the new vocabulary for this lesson, ask them to refer to the class display that you started in this unit. Ask them to review and reflect on the terms and phrases related to the term *Remainder Theorem* that were added to the display during the lesson.

Independent | 5 min

Exit Ticket

HSA.APR.B.2

Students demonstrate their understanding of the Remainder Theorem by rewriting a polynomial as a product of its linear factors.

Printable

Name: ______ Date: ______ Period: ______

Exit Ticket 2.15

Consider the polynomial function $p(x) = 2x^3 + 5x^2 - 6x - 9$. Use the fact that $p(-4) = -33$, $p(-3) = 0$, $p(-2) = 7$, and $p(-1) = 0$ to write $p(x)$ as the product of linear factors. Show your thinking.

Sample response: $p(x) = (x + 3)\ (x + 1)\ (2x - 3)$

	$2x^2$	$-x$	-3
x	$2x^3$	$-x^2$	$-3x$
3	$6x^2$	$-3x$	-9

or

$$\begin{array}{r|l} & 2x^2 - x - 3 \\ \hline x + 3 & 2x^3 + 5x^2 - 6x - 9 \\ & -(2x^3 + 6x^2) \\ \hline & -x^2 - 6x \\ & -(-x^2 - 3x) \\ \hline & -3x - 9 \\ & -(-3x - 9) \\ \hline & 0 \end{array}$$

$$\begin{aligned} p(x) &= (x + 3)\ (2x^2 - x - 3) \\ &= (x + 3)(2x - 3)(x + 1) \end{aligned}$$

Self-Assess

? 1 I don't really get it | 2 I'm starting to get it | 3 I got it

a. I understand the Remainder Theorem and why it is true.
1 2 3

b. I know how to apply the Remainder Theorem to determine whether $(x - a)$ is a factor of polynomial $p(x)$.
1 2 3

Success looks like . . .

» **Goal:** Comprehending that, for a polynomial $p(x)$ and a number a, the remainder on division by $x - a$ is $p(a)$.

» **Goal:** Comprehending that, for a polynomial $p(x)$, $x - a$ is a factor if $p(a) = 0$ and, conversely, that $p(a) = 0$, if $x - a$ is a factor.

✓ Using one of the known zeros to divide the polynomial by a known factor.

Suggested next steps

If students do not recognize the factors of $p(x)$ based on the information in the problem, consider:

- Reviewing what the Remainder Theorem represents when $p(a) = 0$ in Activity 1.
- Assigning Practice Problem 2.
- Asking, "What is the value of the function when the function is evaluated at a zero?"

If students cannot rewrite the polynomial as a product of linear factors, consider:

- Reviewing the example shared in the Summary.
- Assigning Practice Problem 1.
- Asking, "How can you use known linear factors to determine other linear factors of a polynomial function?"

Professional Learning

This professional learning moment is designed to be completed independently or collaboratively with your fellow mathematics educators. Prompts are provided so that you can reflect on this lesson before moving on to the next lesson.

Points to Ponder . . .

- What worked and didn't work today? How was Activity 1 on discovering the Remainder Theorem similar to or different from earlier lessons on polynomial long division?
- Thinking about the questions you asked students today and what the students said or did as a result of the questions, which question was the most effective? What might you change the next time you teach this lesson?

Practice

Independent

Practice

Name: Date: Period:

1. The polynomial function $f(x) = x^3 - 2x^2 - 5x + 6$ has the solutions $f(0) = 6$, $f(2) = -4$, $f(-2) = 0$, $f(3) = 0$, $f(-1) = 8$, $f(1) = 0$. Rewrite $f(x)$ as a product of linear factors.
$f(x) = (x - 1)(x + 2)(x - 3)$

2. Select *all* the polynomials that have $x - 4$ as a factor. Explain or show your thinking.
Sample response:
A. $x^3 - 13x - 12$ — $(4)^3 - 13(4) - 12 = 0$
B. $x^3 + 8x^2 + 19x + 12$ — $(4)^3 + 8(4)^2 + 19(4) + 12 = 280$
C. $x^3 + 6x + 5x - 12$ — $(4)^3 + 6(4) + 5(4) - 12 = 96$
D. $x^3 - x^2 - 10x - 8$ — $(4)^3 - (4)^2 - 10(4) - 8 = 0$
E. $x^2 - 4$ — $(4)^2 - 4 = 12$

3. Long division is used here to divide the polynomial function $p(x) = x^3 + 7x^2 - 20x - 110$ by $x - 5$ and $x + 5$.

$$\begin{array}{r} x^2 + 12x + 40 \\ x - 5\,\overline{)\,x^3 + 7x^2 - 20x - 110} \\ -\,(x^3 - 5x^2) \\ \hline 12x^2 - 20x \\ -\,(12x^2 - 60x) \\ \hline 40x - 110 \\ -(40x - 200) \\ \hline 90 \end{array} \qquad \begin{array}{r} x^2 + 2x - 30 \\ x + 5\,\overline{)\,x^3 + 7x^2 - 20x - 110} \\ -\,(x^3 + 5x^2) \\ \hline 2x^2 - 20x \\ -\,(2x^2 + 10x) \\ \hline 30x + 110 \\ -(-30x - 150) \\ \hline 40 \end{array}$$

a. Evaluate $p(-5)$. **40**

b. Evaluate $p(5)$. **90**

Name: Date: Period:

Practice

4. Which polynomial function has zeros of $x = 5, \frac{2}{3}, -7$?
A. $f(x) = (x + 5)(2x + 3)(x - 7)$
B. $f(x) = (x + 5)(3x + 2)(x - 7)$
C. $f(x) = (x - 5)(2x - 3)(x + 7)$
D. $f(x) = (x - 5)(3x - 2)(x + 7)$

5. The polynomial function $q(x) = 3x^4 + 8x^3 - 13x^2 - 22x + 24$ has known factors $(x + 3)$ and $(x + 2)$. Rewrite $q(x)$ as the product of linear factors.
$q(x) = (x + 3)(x + 2)(x - 1)(3x - 4)$

6. Complete the following tables and determine whether the values represent proportional relationships. Explain your thinking.

x	y
1	$\frac{1}{4}$
12	3
20	5
40	10

The table represents a proportional relationship because y is 4 times less than x and y increases as x increases.

x	y
1	4
12	$\frac{1}{3}$
20	$\frac{1}{5}$
40	$\frac{1}{10}$

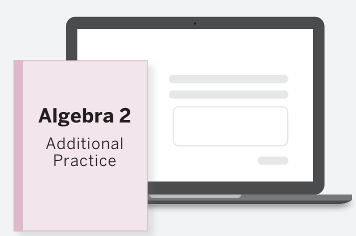
The table does not represent a proportional relationship because $x \cdot y = 4$ and y decreases as x increases.

Practice Problem Analysis

Type	Problem	Refer to	Standard(s)	DOK
On-lesson	1	Activity 1	HSA.APR.B.2	2
	2	Activity 1	HSA.APR.B.2	2
	3	Activity 1	HSA.APR.B.2	2
Spiral	4	Unit 2 Lesson	HSA.APR.B	1
	5	Unit 2 Lesson	HSA.APR.B.3	2
Formative	6	Unit 2 Lesson 16	HSA.CED.A.2	2

Power-up: If students need additional support with the key prerequisite concept or skill this problem addresses, consider assigning the Power-up in the next lesson.

Additional Practice Available



For students who need additional practice in this lesson, assign the **Algebra 2 Additional Practice**.

Will You Escape?

Let's use what we know about polynomial and rational functions to solve some problems.

Focus

Goals

1. Apply knowledge of polynomial and rational functions to identify key features of their graphs.
2. Perform arithmetic operations with polynomial and rational expressions and identify properties of their equations.
3. Rewrite and solve rational equations.

Coherence

Today

Students use their understanding of polynomial and rational functions to sketch graphs, analyze the structure of their equations, and rewrite them to solve problems.

Previously

In Lesson 22, students determined what made an equation an identity and proved some common identities.

Coming Soon

In Unit 3, students will explore function transformations and applications of conic sections.

Rigor

- Students **apply** their knowledge about the structure of a polynomial to sketch its graph.
- Students **apply** their knowledge about asymptotes of a rational function to sketch its graph.
- Students **apply** their knowledge of polynomial long division and the Remainder Theorem to determine the zeros and missing terms of a polynomial.
- Students **apply** their knowledge of rational equations to solve for missing terms in a rational expression.

Standards

Addressing

HSF.IF.C.7.C
Graph polynomial functions, identifying zeros when suitable factorizations are available, and showing end behavior.

Also Addressing: **HSF.IF.C.7.D, HSA.APR.A.1, HSA.SPR.B.2, HSA.REI.A.2, HSA.APR.B, HSA.APR.D**

Pacing Guide

Suggested Total Lesson Time ~**50 min**

Warm-up	Activity 1	Activity 2	Activity 3	Summary	Exit Ticket
5 min	10 min	10 min	15 min	5 min	5 min
Small Groups	Small Groups	Small Groups	Small Groups	Whole Class	Independent
MP7	MP7	MP2	MP7		
HSF.IF.C.7.C	HSA.APR.B, HSF.IF.C.7.D	HSA.APR.A.1, HSA.APR.B.2	HSA.REI.A.2, HSA.APR.D.6		HSA.APR.B, HSA.APR.D

Amps powered by desmos | **Activity and Presentation Slides**

For a digitally interactive experience of this lesson, log in to Amplify Math at **learning.amplify.com**.

Practice | Independent

Materials

- Exit Ticket
- Additional Practice
- Anchor Chart PDF, *Remainder Theorem*
- graphing technology

Math Language Development

Review words

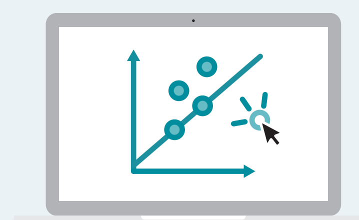
- *end behavior*
- *horizontal asymptote*
- *multiplicity*
- *rational function*
- *Remainder Theorem*
- *slant asymptote*
- *vertical asymptote*
- *zeros*

Amps | **Featured Activity**

Activity 1

Interactive Graphs

Students try to guess the hidden graph by changing the parameters to see how the structure of a function affects its graph.



Amps POWERED BY desmos

Building Math Identity and Community

Connecting to Mathematical Practices

Students may struggle to identify the correct graph based on all the clues if they do not remember the characteristics of polynomial and rational functions. Explain to students that they need to step back and make use of prior knowledge **(MP7)** to connect the clues to the correct graph. Suggest students persist when frustrated by making a list of characteristics to reference (end behavior, zeros, multiplicity, asymptotes, etc.).

Modifications to Pacing

You may want to consider these additional modifications if you are short on time.

- In **Activity 1**, eliminate one of the graph options in Problems 1 and 2.
- Provide one of the missing values in Problems 1 and 2 of **Activity 3**.
- Provide a value for the remainder in Problem 3 of **Activity 3**.

Small Groups | 5 min

Warm-up What Can It Be?

MP7
HSF.IF.C.7.C

Students create sketches of polynomial and rational functions with specific features to reinforce their understanding of the shapes of these graphs **(MP7).**

Name: ______ Date: ______ Period: ______

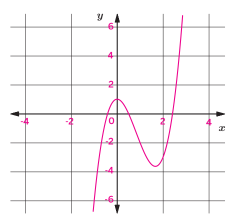
Unit 2 | Lesson 23 – Capstone

Will You Escape?

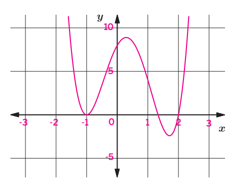
Let's use what we know about polynomial and rational functions to solve some problems.

Warm-up What Can It Be?

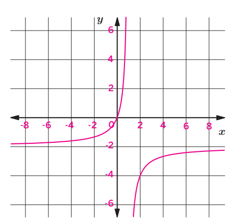
1. Sketch a polynomial function with odd degree that has a local maximum at (0, 1).



2. Sketch a polynomial function with even degree that has at least one zero with a multiplicity of 2.



3. Sketch a rational function with at least one vertical asymptote and one horizontal asymptote.



Log in to Amplify Math to complete this lesson online.

Lesson 23 Will You Escape? 263

1 Launch

Provide graphing technology and tell students there are many possible responses.

2 Monitor

Help students get started by having them explain what they know about each graph from the given description.

Look for points of confusion:

- **Not connecting a polynomial's end behavior to the degree of its leading term.** Have students use graphing technology to analyze the graphs of different polynomials where the degree is even or odd.
- **Not relating the end behavior of a rational function to its horizontal asymptote.** Ask students to describe different types of end behavior for rational functions.

Look for productive strategies:

- Sketching the end behavior and x-intercepts of a polynomial function before using the multiplicity and local extrema to complete the graph.
- Writing equations that satisfy the given requirements for each graph.

3 Connect

Use the ***Gallery Tour*** routine to display student graphs.

Highlight how different functions can share common characteristics and note the similarities in student graphs for the polynomial and rational functions.

Math Language Development

MLR7: Compare and Connect

During the ***Gallery Tour*** routine, have pairs of students identify what is the same and what is different between the various graphs. Display the following questions for students to refer to as they participate in the ***Gallery Tour***:

- "What do you notice about the functions with odd/even degrees?"
- "What do you notice about the functions with horizontal and vertical asymptotes?"

Consider asking students to provide sketches of the graphs before using graphing technology to verify their sketches.

Power-up

To power up students' understanding of asymptotes, ask:

What are the asymptotes of this rational function? Display the graph of the following equation:

$$y = \frac{x^2}{(x-1)(x+2)}$$

Vertical asymptotes are $x = 1$, $x = -2$. Horizontal asymptote is $y = 1$.

Use: Before the Warm-Up

Informed by: Performance on Lesson 22, Practice Problem 6

Small Groups | 10 min

Activity 1 Guess the Graph

MP7
HSA.APR.B, HSF.IF.C.7.D

Students analyze graphs with missing information to think critically about the key features of a polynomial or rational function.

Amps Featured Activity Interactive Graphs

Activity 1 Guess the Graph

Oh no! You and your classmates have been accidentally locked in the classroom. Can you find the way out? The following activities will give you clues to solve tasks and find the "key" to get out.

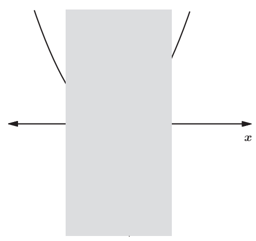
1. Figure out what this mysterious graph looks like to determine the function it represents and unlock Activity 2.

   Here are three clues. Determine the mystery function and explain your thinking.

   Clue 1: The mystery function is a polynomial.

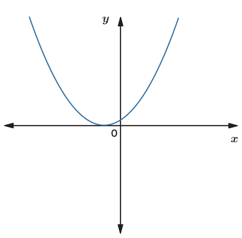
   Clue 2: $x = -1$ is a zero of the mystery function with a multiplicity of 2.

   Clue 3: The mystery function has a positive y-intercept.

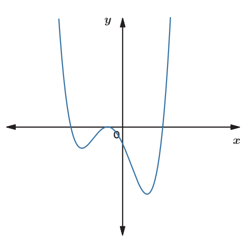


   Consider the following candidates:

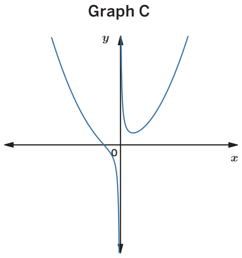
Graph A



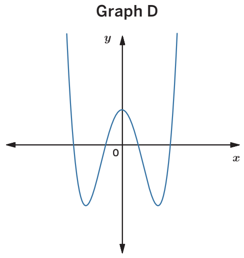
Graph B



Graph C



Graph D



   The mystery function is represented in Graph A.
   Sample response: Graph C is a rational function, Graph D has a zero of $x = -1$ with a multiplicity of 1, and Graph B has a negative y-intercept.

1 Launch

Students work independently and then discuss their thinking in small groups. Provide access to graphing technology.

2 Monitor

Help students get started by asking, "What does the end behavior of each mystery graph tell you about the characteristics of that function?"

Look for points of confusion:

- **Misunderstanding how the degrees of the polynomials in a rational function are related to its end behavior.** Activate prior knowledge by asking students to use graphing technology to review examples of the following different cases:

$$y = \frac{1}{x-3},\ y = \frac{2x^2 - 3x}{x^2 - 1},\ y = \frac{x^2 - 6x}{2x + 1}$$

Look for productive strategies:

- Using each clue to eliminate graphs that do not meet the requirements.

Activity 1 continued ›

Differentiated Support

Accessibility: *Optimize Access to Technology*

Have students use the Amps slides for this activity, in which they can try to guess the hidden graph by changing the parameters to see how the structure of a function affects its graph.

Math Language Development

MLR2: Collect and Display

After students complete Problem 2, amplify language students use to justify their response. Emphasize the language students have developed over the course of the unit and how it has helped them communicate and convey their mathematical thinking clearly to their peers.

English Learners: Use gestures and pointing to highlight all of the language and diagrams that have been added to the class displays during the unit. Give students time to silently reflect on their own personal development of the new mathematical language.

Activity 1 Guess the Graph (continued)

MP7
HSF.IF.C.7

Students analyze graphs with missing information to think critically about the key features of a polynomial or rational function.

Name: Date: Period:

Activity 1 Guess the Graph (continued)

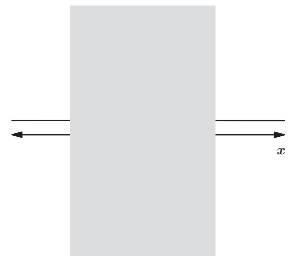
2. Figure out what this new mysterious graph looks like to determine the function it represents and unlock Activity 2.

Here are three clues. Determine the mystery function and explain your thinking.

Clue 1: The mystery function has at least one vertical asymptote.

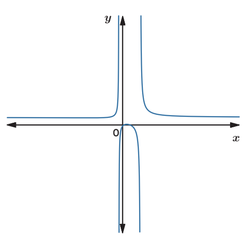
Clue 2: The degree of the polynomial in the denominator of the mystery function is not 1.

Clue 3: The mystery function has no x-intercept.

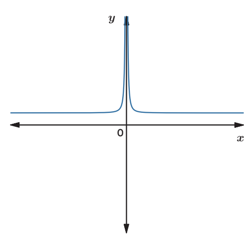


Consider the following candidates:

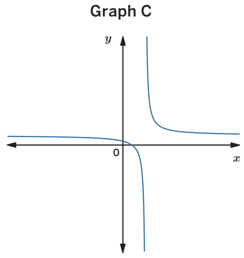
Graph A



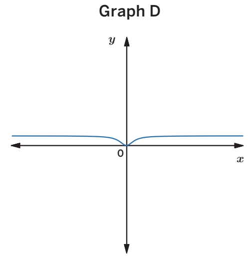
Graph B



Graph C



Graph D



The mystery function is represented in Graph B.

Sample response: Graph D has no vertical asymptotes, the degree of the polynomial in the denominator of Graph C is 1, and Graph A has x and y-intercepts.

3 Connect

Display the mystery graphs for Problems 1 and 2.

Have groups of students share their reasons why the other 3 graphs do not satisfy the clues.

Highlight that the structure of polynomial and rational functions can be used to identify the different types of end behavior **(MP7)**.

Ask, "Can a rational function exhibit the same type of end behavior as a polynomial function?" Yes. Sample response: If the degree of the polynomial in the numerator of a rational function is 2 or more than the degree of the polynomial in the denominator, then the end behavior can approach positive or negative infinity.

Small Groups | 10 min

MP2
HSA.APR.A.1, HSA.APR.B.2

Activity 2 Crack the Code

Students apply their knowledge of the Remainder Theorem to reason algebraically about problems involving operations on polynomials **(MP2).**

Activity 2 Crack the Code

Use your two answers from the previous activity to choose two polynomials to use in this task.

$A(x) = x^2 - 4x - 49$ $B(x) = x - 4$ $C(x) = 2x^3 - 3x^2 - 23x + 12$ $D(x) = x + 1$

1. Determine the product of the two polynomials.

$x^3 - 8x^2 - 33x + 196$

2. Use long division to determine the zeros of $C(x)$, given that $(2x - 1)$ is a known factor. Show your thinking.

$x = \frac{1}{2}, -3, 4$

$$\begin{array}{r} x^2 - x - 12 \\ 2x - 1 \overline{\smash{)}\, 2x^3 - 3x^2 - 23x + 12} \\ \underline{-(2x^3 - x^2)} \\ -2x^2 - 23x \\ \underline{-(-2x^2 + x)} \\ -24x + 12 \\ \underline{-(-24x + 12)} \\ 0 \end{array}$$

$C(x) = (x^2 - x - 12)(2x - 1) = (x - 4)(x + 3)(2x - 1)$

1 Launch

Have students work independently for 5 minutes before sharing their thinking with their group.

2 Monitor

Help students get started by activating prior knowledge and asking, "How can you determine a polynomial's factors?" Sample response: If $P(x)$ is divided by $(x - a)$ and the remainder is 0, or $P(a) = 0$, then $(x - a)$ is a factor of $P(x)$.

Look for points of confusion:

- **Incorrectly interpreting the Remainder Theorem.** Display the Anchor Chart PDF, *Remainder Theorem*.

Look for productive strategies:

- Verifying the zeros of a polynomial by graphing it.
- Applying the Remainder Theorem to more efficiently solve problems.

Activity 2 continued >

Differentiated Support

Accessibility: *Activate Prior Knowledge*

During the Monitor, demonstrate or model for students how to determine a polynomial's factors, then display the Anchor Chart PDF, *Remainder Theorem* and instruct students to refer to it during the activity.

Accessibility: *Vary Demands to Optimize Challenge*

Consider providing students with a checklist to help students understand the task, plan the task, and ensure that each problem is solved before unlocking the secret code in Problem 5.

Small Groups | 10 min

Activity 2 Crack the Code (continued)

MP2
HSA.APR.A.1, HSA.APR.B.2

Students apply their knowledge of the Remainder Theorem to reason algebraically about problems involving operations on polynomials **(MP2).**

Name: ______ Date: ______ Period: ______

Activity 2 Crack the Code (continued)

3. What is the remainder when $C(x)$ is divided by $(x + 1)$? Show your thinking.

30

$$\begin{aligned} C(-1) &= 2(-1)^3 - 3(-1)^2 - 23(-1) + 12 \\ &= -2 - 3 + 23 + 12 \\ &= 30 \end{aligned}$$

4. If k is a constant, what is the value of k such that the remainder of $k^2x^3 - 6kx + 9$ divided by $(x - 1)$ is 7? Show your thinking.

3

$$\begin{aligned} k^2(1)^3 - 6k(1) + 9 &= 7 \\ k^2 - 6k + 9 &= 0 \\ (k-3)^2 &= 0 \\ k &= 3 \end{aligned}$$

5. Use your responses in Problems 1–4 to determine the secret code that unlocks Activity 3.

a. Identify the constant term from the resulting polynomial in Problem 1 and record the digits in the first three spaces of the following code box. Be sure to include negative signs when appropriate. **196**

b. Determine the product of the values of the zeros in Problem 2 and add the remainder from Problem 3. Record the digits in the next two spaces of the code box. **24**

c. Write the value of k from Problem 4 in the last space of the code box. **3**

1	9	6	2	4	3

Lesson 23 Will You Escape? 267

3 Connect

Display Problem 4.

Have groups of students share their strategies to solve for k.

Highlight that we can use the Remainder Theorem to evaluate a polynomial at a specific input or determine whether a divisor is also a factor of the polynomial.

Ask, "What is the quotient when $\frac{P(x)}{(x-1)}$?" $9x^2 + 9x - 9$

Small Groups | 15 min

Activity 3 Mystery Message

MP7
HSA.REI.A.2, HSA.APR.D.6

Students apply their knowledge of rational equations to solve for missing terms in a rational expression. In addition, students use the structure of a rational function to identify asymptotes **(MP7).**

Activity 3 Mystery Message

The secret code from the previous activity will help you solve Problems 1 and 2. Complete all the problems to decipher the message at the end of this activity.

1. Use the first three digits of the secret code to complete the following rational equation so that the solution is an even number greater than 0.

$$\frac{\boxed{9}a - 6}{a + \boxed{1}} = \boxed{6}$$

$a = 4$

2. Use the last three digits of the secret code to complete the following rational equation so that the solution is an integer greater than 2.

$$\frac{\boxed{4}b}{\boxed{3} + b} = \frac{\boxed{2}}{4 - b}$$

$b = 3$

3. Write a rational function, f, with a slant asymptote of $y = 3x - 4$ and a vertical asymptote of $x = -6$. Express your equation in the form $\frac{A(x)}{B(x)}$, where $B(x) = x + 6$. Add the digits of the coefficient of the linear term in the numerator of your function. You will use this sum in Problem 4.

Coefficient sum: $= 1 + 4 = 5$

$y = 3x - 4 + \frac{r}{x+6}$ (r can be any constant)
If $r = 2$, then $f(x) = \frac{3x^2 - 14x - 22}{x + 6}$

1 Launch

Provide graphing technology. Students should work independently on Problem 2 for 2 minutes, and then switch papers with another group member to complete the problem. They work in small groups on the remaining problems.

2 Monitor

Help students get started by asking, "What methods did you use in this unit to solve equations?"

Look for points of confusion:

- **Not creating a rational function with the correct asymptote.** Remind students that they can check their work by graphing the rational function and the slant asymptote on the same coordinate grid.

Look for productive strategies:

- Graphing the expressions in the equations to determine the intersection point.

Activity 3 continued >

Small Groups | 15 min

Activity 3 Mystery Message (continued)

MP7
HSA.REI.A.2, HSA.APR.D.6

Students apply their knowledge of rational equations to solve for missing terms in a rational expression. In addition, students use the structure of a rational function to identify asymptotes **(MP7).**

Name: ______ Date: ______ Period: ______

Activity 3 Mystery Message (continued)

4. The solutions from Problems 1–3 (a, b, and the sum) are the side lengths of a common shape. Fill in the blanks to describe the shape. The symbols below each letter are part of a substitution cipher, in which there is a unique symbol for each letter of the alphabet.

R I G H T   T R I A N G L E

5. Use the information from Problem 4 to decipher the following message and unlock the door!

O N E   K E Y   T O   M A T H

I S   A   G R O W T H   M I N D S E T

Are you ready for more?

The following identity can be used to generate Pythagorean triples, where x and y are both positive and $x \neq y$: $(x^2 + y^2)^2 = (x^2 - y^2)^2 + (2xy)^2$.

For example, substituting $x = 2$ and $y = 1$ into the identity produces the triple $3 - 4 - 5$. Determine another solution for x and y that corresponds to the triple $6 - 8 - 10$.

$x = 3$, $y = 1$

STOP

3 Connect

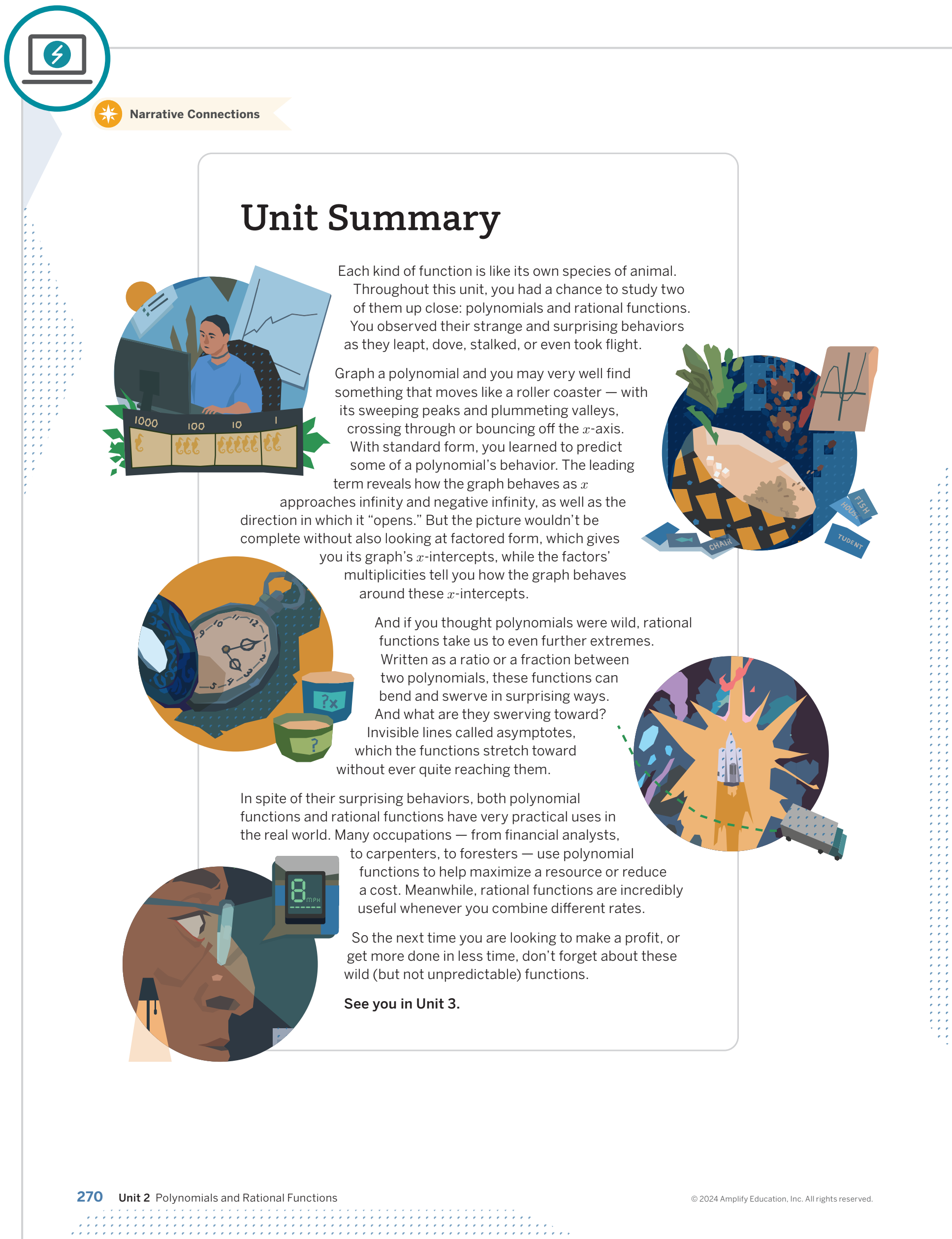
Display a student's response to Problem 3 in addition to a graph of the student's function and asymptote.

Have students share their functions for Problem 3. Select and sequence those with different remainders.

Highlight the similarities between the different possible functions.

Unit Summary

Review and synthesize how to rewrite and interpret the structure of polynomial and rational functions to identify key features of their graphs, perform arithmetic operations on polynomials, and solve rational equations.



Narrative Connections

Unit Summary

Each kind of function is like its own species of animal. Throughout this unit, you had a chance to study two of them up close: polynomials and rational functions. You observed their strange and surprising behaviors as they leapt, dove, stalked, or even took flight.

Graph a polynomial and you may very well find something that moves like a roller coaster — with its sweeping peaks and plummeting valleys, crossing through or bouncing off the x-axis. With standard form, you learned to predict some of a polynomial's behavior. The leading term reveals how the graph behaves as x approaches infinity and negative infinity, as well as the direction in which it "opens." But the picture wouldn't be complete without also looking at factored form, which gives you its graph's x-intercepts, while the factors' multiplicities tell you how the graph behaves around these x-intercepts.

And if you thought polynomials were wild, rational functions take us to even further extremes. Written as a ratio or a fraction between two polynomials, these functions can bend and swerve in surprising ways. And what are they swerving toward? Invisible lines called asymptotes, which the functions stretch toward without ever quite reaching them.

In spite of their surprising behaviors, both polynomial functions and rational functions have very practical uses in the real world. Many occupations — from financial analysts, to carpenters, to foresters — use polynomial functions to help maximize a resource or reduce a cost. Meanwhile, rational functions are incredibly useful whenever you combine different rates.

So the next time you are looking to make a profit, or get more done in less time, don't forget about these wild (but not unpredictable) functions.

See you in Unit 3.

270 Unit 2 Polynomials and Rational Functions

© 2024 Amplify Education, Inc. All rights reserved.

Narrative Connections

Read the narrative aloud as a class or have students read it individually.

Synthesize

Display the hidden message in Activity 3.

Highlight that polynomial and rational functions are related and that the methods used to analyze and apply their characteristics to solve problems are similar.

Ask, "How did you apply the knowledge you have learned over the course of this unit to today's activities?"

Reflect

After synthesizing the concepts of this unit, allow students a few moments for reflection around the concepts of the unit. To help them engage in meaningful reflection, consider asking:

- "Did anything surprise you while reading the narratives of this unit?"
- "Is there anything you would like to learn more about these topics? What are some steps you can take to learn more?"

Independent | 5 min

Exit Ticket

HSA.APR.B, HSA.APR.D

Students demonstrate their understanding of polynomial and rational functions to create equations that have specific features.

Printable

Name: ____________ Date: ____________ Period: ____________

Exit Ticket 2.23

1. Write an equation for a polynomial function in standard form with all of the following features:
   - The polynomial has a negative y-intercept.
   - The graph has one local maximum and one local minimum.
   - The multiplicity of one of the factors is even.

   Sample response: $y = 2x^3 - 9x^2 + 12x - 4$

2. Divide the polynomial function from Problem 1 by another polynomial function to produce a rational function with at least one asymptote. State the equations of all the asymptotes.

   Sample response: $y = \frac{2x^3 - 9x^2 + 12x - 4}{(x-1)(x+2)^2}$. **Vertical asymptotes at** $x = 1$, $x = -2$; **horizontal asymptote at** $y = 2$.

Self-Assess

? 1 I don't really get it — 2 I'm starting to get it — 3 I got it

a. I can identify features of polynomials and their graphs using their standard and factored forms.
1 2 3

b. I can identify an asymptote from a graph or an equation of a rational function.
1 2 3

Success looks like . . .

» **Goal:** Applying knowledge of polynomial and rational functions to identify key features of their graphs.

  ✓ Writing an equation to represent a polynomial function in Problem 1.

» **Goal:** Performing arithmetic operations with polynomial and rational expressions and identifying properties of their equations.

» **Goal:** Rewriting and solving rational equations.

  ✓ Writing an equation to represent a rational function in Problem 2.

Suggested next steps

If students cannot solve Problem 1, consider:

- Reviewing strategies from Activity 1.

If students cannot solve Problem 2, consider:

- Reviewing strategies from Activity 2.

Professional Learning

This professional learning moment is designed to be completed independently or collaboratively with your fellow mathematics educators. Prompts are provided so that you can reflect on this lesson before moving on to the next lesson.

Points to Ponder . . .

- What worked and didn't work today? What challenges did students encounter as they worked on guessing the mystery graphs?
- Did students find Activity 1, Activity 2 or Activity 3 more engaging today? Why do you think that is? What might you change the next time you teach this lesson?

Practice

Independent

Name: Date: Period:

Practice

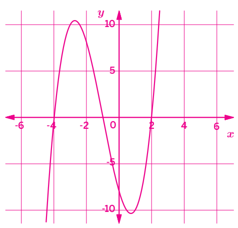
1. The polynomial function $p(x) = x^3 + 3x^2 - 6x - 8$ has a known factor of $x + 4$.

   a. Rewrite $p(x)$ as the product of linear factors. Show your thinking.

   $p(x) = (x+4) \cdot (x^2 - x - 2) = (x+4)(x-2)(x+1)$

   $$\begin{array}{r|l} & x^2 - x - 2 \\ x+4 & x^3 + 3x^2 - 6x - 8 \\ & -(x^3 + 4x^2) \\ & -x^2 - 6x \\ & -(-x^2 - 4x) \\ & -2x - 8 \\ & -(-2x - 8) \\ & 0 \end{array}$$

   b. Sketch the graph of the function.



2. The graph of a polynomial $f(x) = (5x - 3)(x + 4)(x + a)$ has the zeros at $x = -4$, $\frac{3}{5}$, and 6. What is the value of a? $a = -6$

3. Use the Remainder Theorem to determine whether $x - 3$ is a factor of $S(x) = x^4 + 2x^3 - 10x^2 - 18x + 9$. Explain your thinking.

   Yes; Sample response: $S(3) = 0$, so $x - 3$ is a factor of $S(x)$.

Practice

Name: Date: Period:

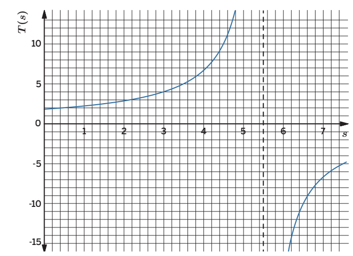
4. Andre is planning a kayak trip. He determines an expression for the time $T(s)$, in hours, it takes him to paddle 10 km upstream in terms of s, the speed of the current in kilometers per hour. This is the graph Andre produces if he allows s to take on any value between 0 and 7.5.

   a. What would be a more appropriate domain for Andre to use instead?

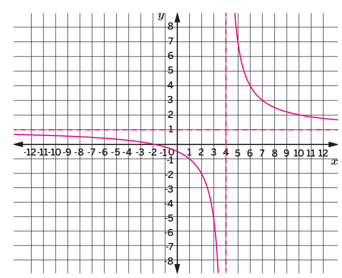
   $0 < s < 5.5$

   b. What is the approximate speed of the current if his trip takes 6 hours?

   3.8 kph



5. Sketch the graph of the rational function, $h(x) = \frac{x+2}{x-4}$, including any asymptotes.



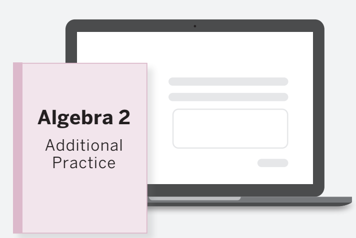
6. Han is asked to solve the equation $\frac{x^2}{x-7} = \frac{49}{x-7}$. He multiplies both sides of the equation by $x - 7$, which results in $x^2 = 49$. He concludes that the solution must be $x = \pm 7$. Explain and correct Han's mistake.

   Sample response: Han says one of the answers is $x = 7$, but this is an extraneous solution because it makes the denominator zero. Therefore, x can only equal -7.

Practice Problem Analysis

Type	Problem	Refer to	Standard(s)	DOK
On-lesson	1	Activity 1 and 2	HSA.APR.B, HSA.APR.B.2	2
	2	Activity 1	HSA.APR.B	1
	3	Activity 2	HSA.APR.B.2	2
Spiral	4	Unit 2 Lesson 16	HSA.CED.A.2	1
	5	Unit 2 Lesson 18	HSF.APR.D.6	2
	6	Unit 2 Lesson 21	HSA.REI.A.2	2

Additional Practice Available



For students who need additional practice in this lesson, assign the **Algebra 2 Additional Practice**.